Computer Graphics

- Signal Processing -

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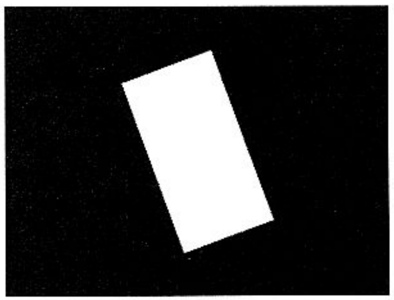
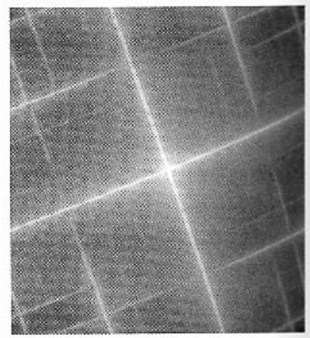
Overview

- **Last time**
  - Texture synthesis
  - Procedural textures
- **Today**
  - Signal Processing
- **Next lecture**
  - Antialiasing

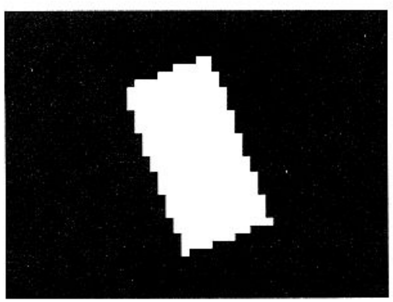
Motivation: Aliasing



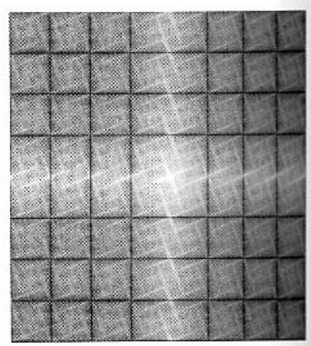
(a) Simulation of a perfect line



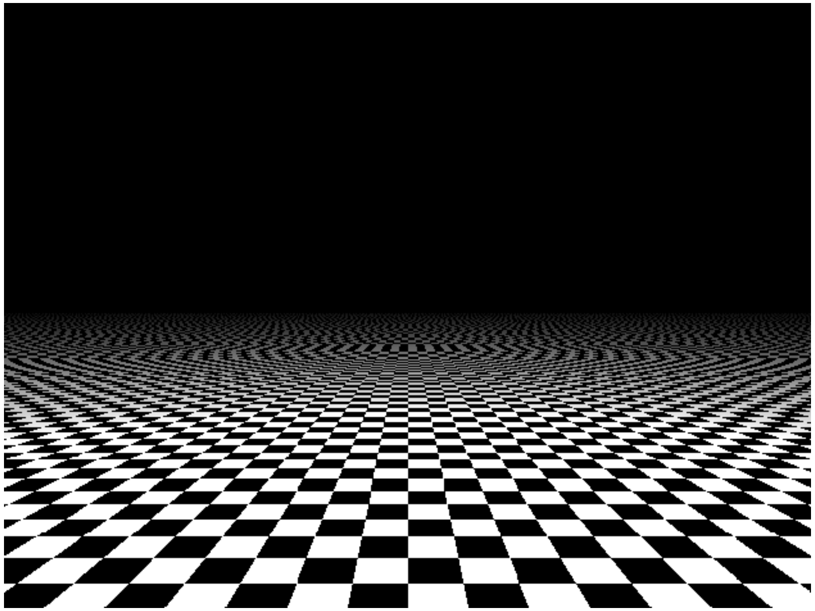
(b) Fourier transform of (a)

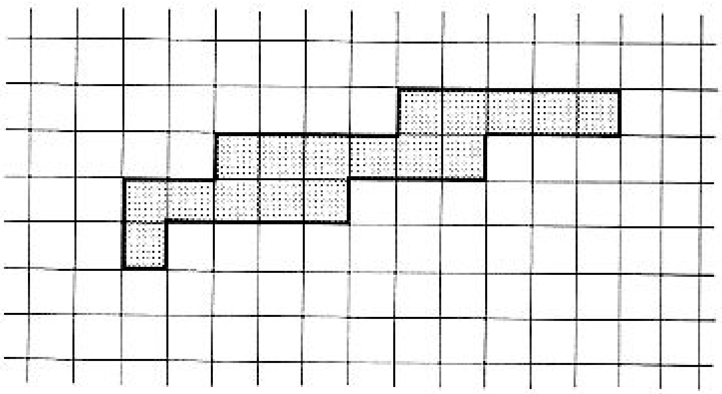


(c) Simulation of a jagged line



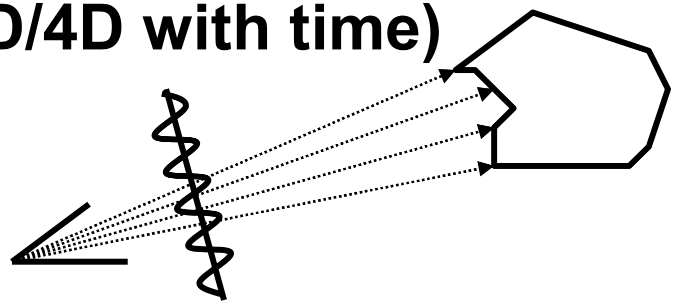
(d) Fourier transform of (c)



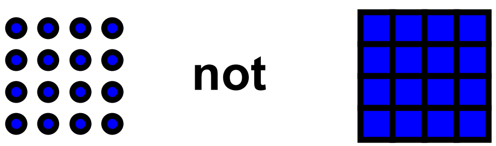


The Digital Dilemma

- **Nature: continuous signal (2D/3D/4D with time)**
  - Defined at every point



- **Acquisition: sampling**
  - Rays, pixel/texel, spectral values, frames, ...
- **Representation: discrete data**
  - Discrete points, discretized values



- **Reconstruction: filtering**
  - Mimic continuous signal
- **Display and perception: faithful**
  - Hopefully similar to the original signal, no artifacts

Sensors

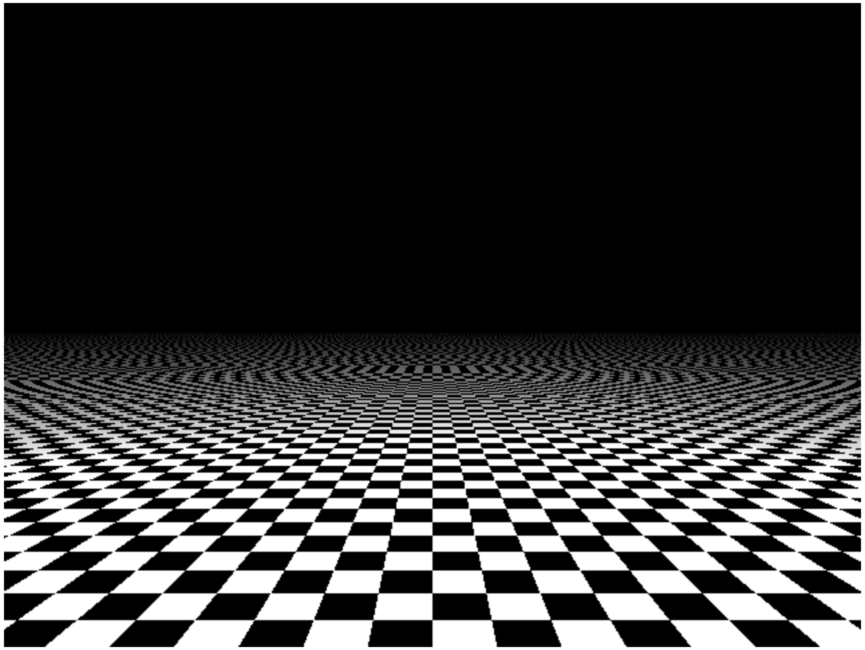
- **Sampling of signals**
  - Conversion of a continuous signal to discrete samples by integrating over the sensor field
  - Required by physical processes

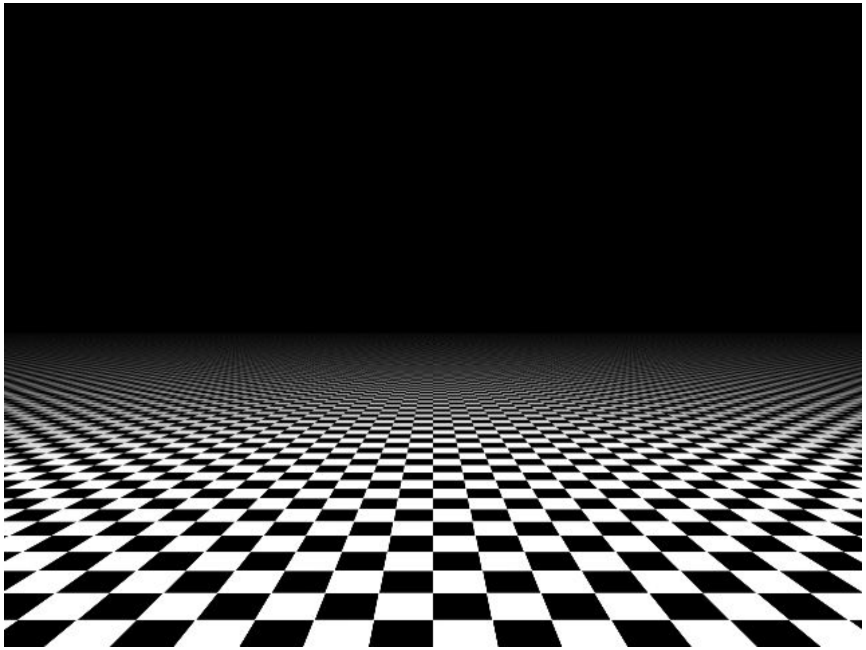
$$R(i, j) = \int_{A_{ij}} E(x, y) P_{ij}(x, y)\, dx\, dy$$

- **Examples**
  - Photo receptors in the retina
  - CCD or CMOS cells in a digital camera
- **Virtual cameras in computer graphics**
  - Integration is too expensive and usually avoided
  - Ray tracing: mathematically ideal point samples
    - Origin of aliasing artifacts !

Aliasing

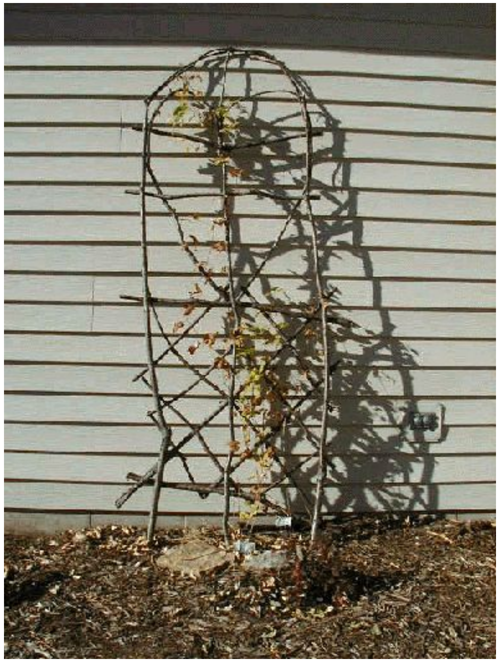
- **Ray tracing**
  - Textured plane with one ray for each pixel (say, at pixel center)
    - No texture filtering: equivalent to modeling with b/w tiles
  - Checkerboard period becomes smaller than two pixels
    - At the Nyquist limit
  - Hits textured plane at only one point, black or white by “chance

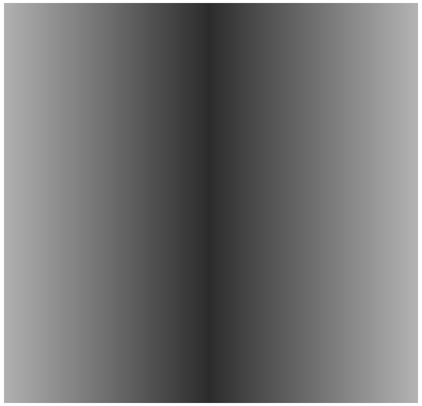




Spatial Frequency

- **Frequency: period length of some structure in an image**
  - Unit [1/pixel]
  - Range: -0.5…0.5 ($-\pi \ldots \pi$)
- **Lowest frequency**
  - Image average
- **Highest frequency: Nyquist limit**
  - In nature: defined by wavelength of light
  - In graphics: defined by image resolution





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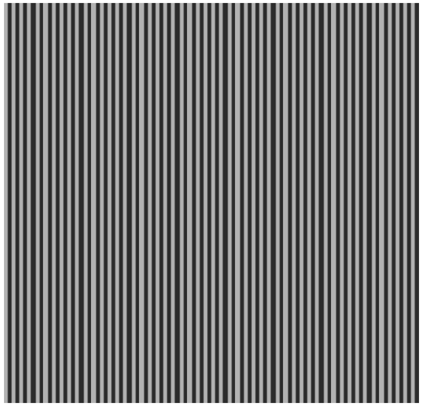
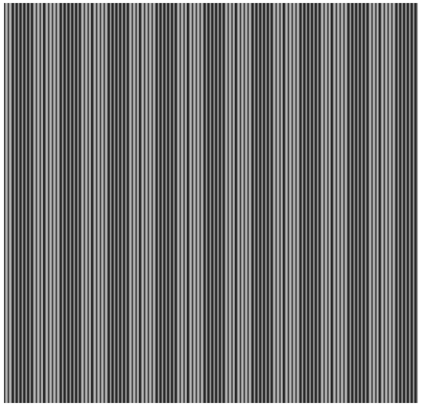
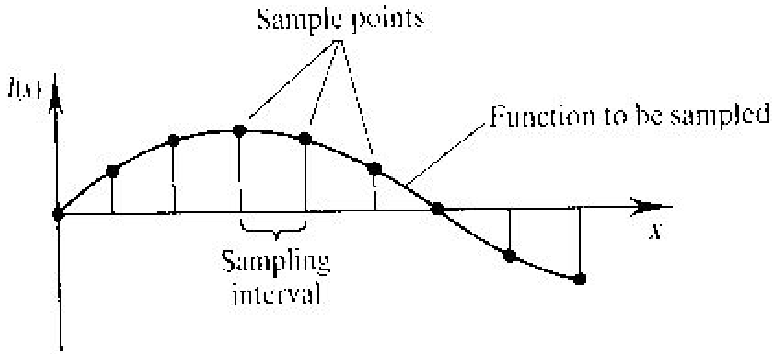


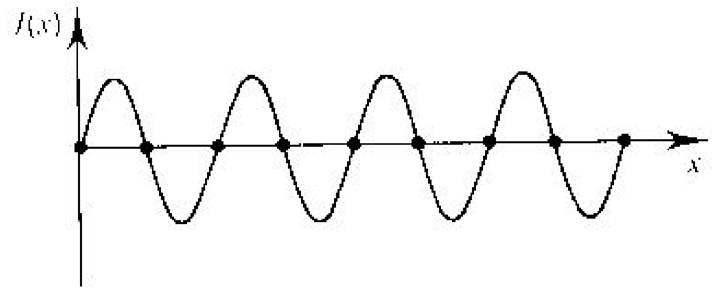


Nyquist Frequency

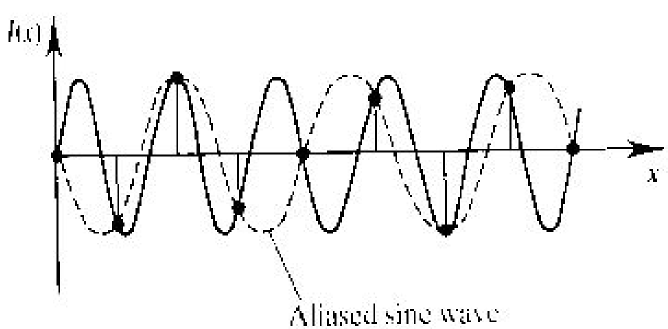
- **Highest (spatial) frequency that can be represented**
- **Determined by image resolution (pixel size)**



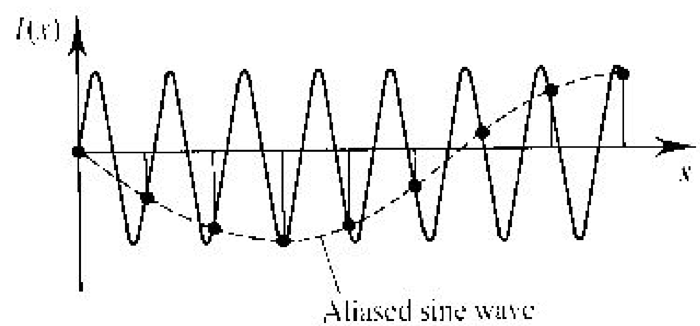
Spatial frequency < Nyquist



Spatial frequency = Nyquist
2 samples / period



Spatial frequency > Nyquist



Spatial frequency >> Nyquist

Fourier Transformation

- **Any continuous function f(x) can be expressed as an integral over sine and cosine waves:**

$$F(k) = F_x[f(x)](k) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx \qquad \text{Analysis}$$

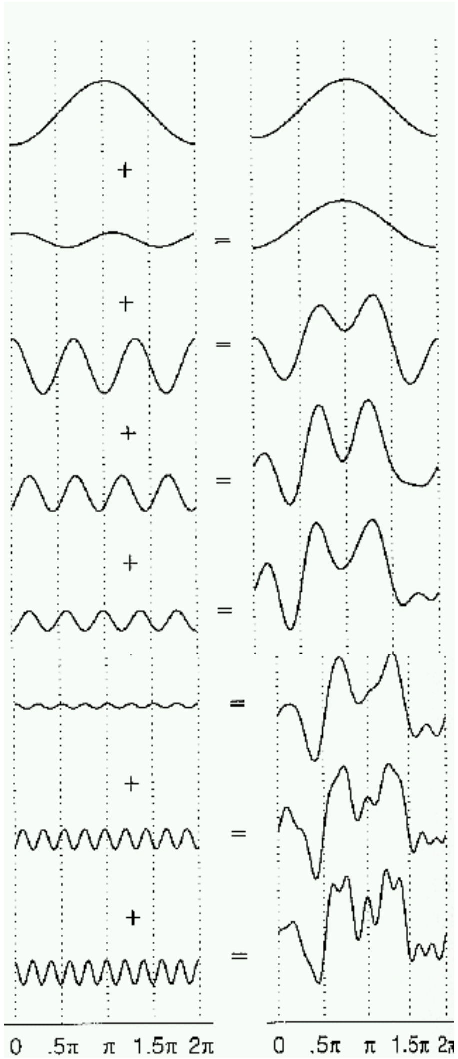
$$f(x) = F_x^{-1}[F(k)](x) = \int_{-\infty}^{\infty} F(k)\, e^{2\pi i k x}\, dk \qquad \text{Synthesis}$$

- **Division into even and odd parts**

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = E(x) + O(x)$$

- **Transform of each part**

$$F[f(x)](k) = \int_{-\infty}^{\infty} E(x)\cos(2\pi k x)\, dx - i\int_{-\infty}^{\infty} O(x)\sin(2\pi k x)\, dx$$



Fourier Transformation

- **Any periodic, continuous function can be expressed as the sum of an (infinite) number of sine or cosine waves:**

  $$f(x)=\Sigma_k\ a_k \sin(2\pi*k*x) + b_k \cos(2\pi*k*x)$$

  - Decomposition of signal into different frequency bands
    - Spectral analysis
  - k: frequency band
    - k=0 mean value
    - k=1 function period, lowest possible frequency
    - k=1.5? not possible, periodic function f(x) = f(x+1)
    - k_{max}? band limit, no higher frequency present in signal
  - a_k,b_k : (real-valued) Fourier coefficients
    - Even function f(x)= f(-x): $a_k = 0$
    - Odd function f(x)= -f(-x): $b_k = 0$

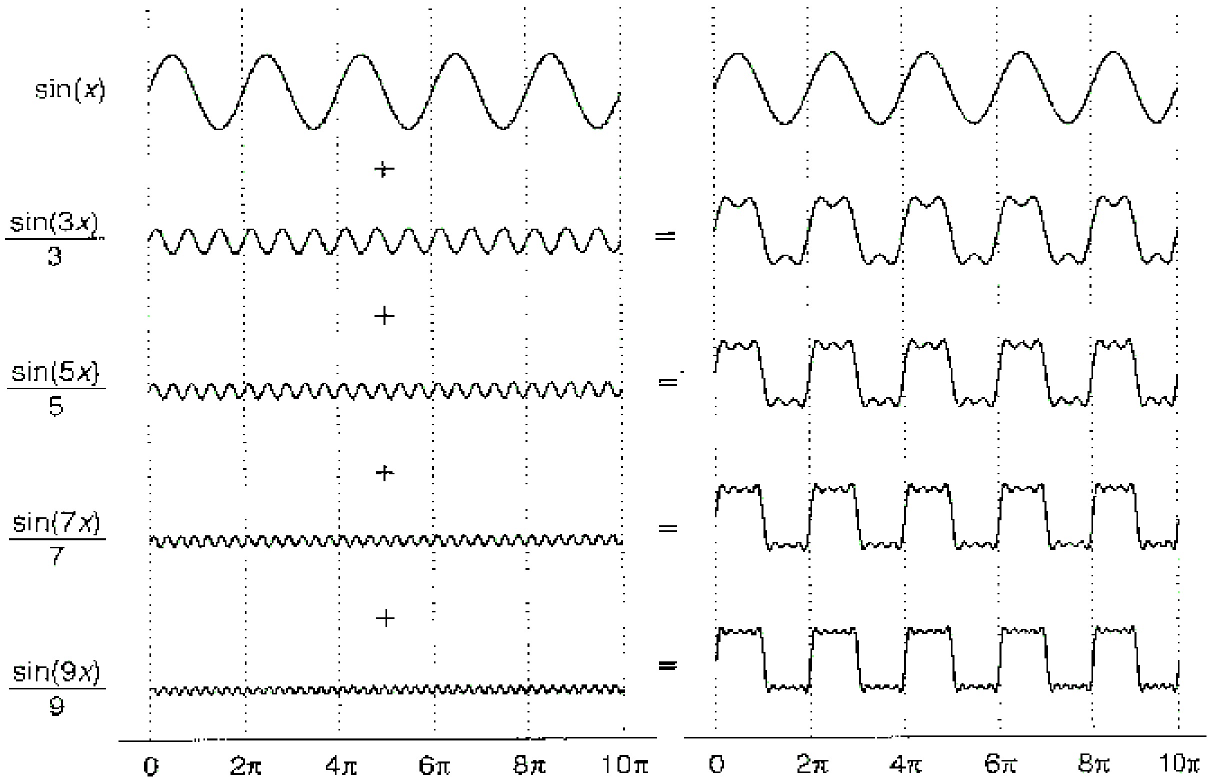
Fourier Synthesis Example

- **Periodic, uneven function: square wave**

$$f(x) = 0.5 \ \forall\ 0 < (x \bmod 2\pi) < \pi$$
$$= -0.5 \ \forall\ \pi < (x \bmod 2\pi) < 2\pi$$

$a_k = \int \sin(k*x) * f(x)\, dx \qquad f(x) = \Sigma_k\, a_k \sin(k*x)$

- $a_0 = 0$
- $a_1 = 1$
- $a_2 = 0$
- $a_3 = 1/3$
- $a_4 = 0$
- $a_5 = 1/5$
- $a_6 = 0$
- $a_7 = 1/7$
- $a_8 = 0$
- $a_9 = 1/9$
- …



Discrete Fourier Transform

- **Equally-spaced function samples**
  - Function values known only at discrete points
    - Physical measurements
    - Pixel positions in an image !
- **Fourier Analysis**

$$a_k = 1/N \sum_i \sin(2\pi k i / N)\; f_i\,, \qquad b_k = 1/N \sum_i \cos(2\pi k i / N)\; f_i$$

  - Sum over all measurement points N
  - k=0,1,2, …, ? Highest possible frequency ?

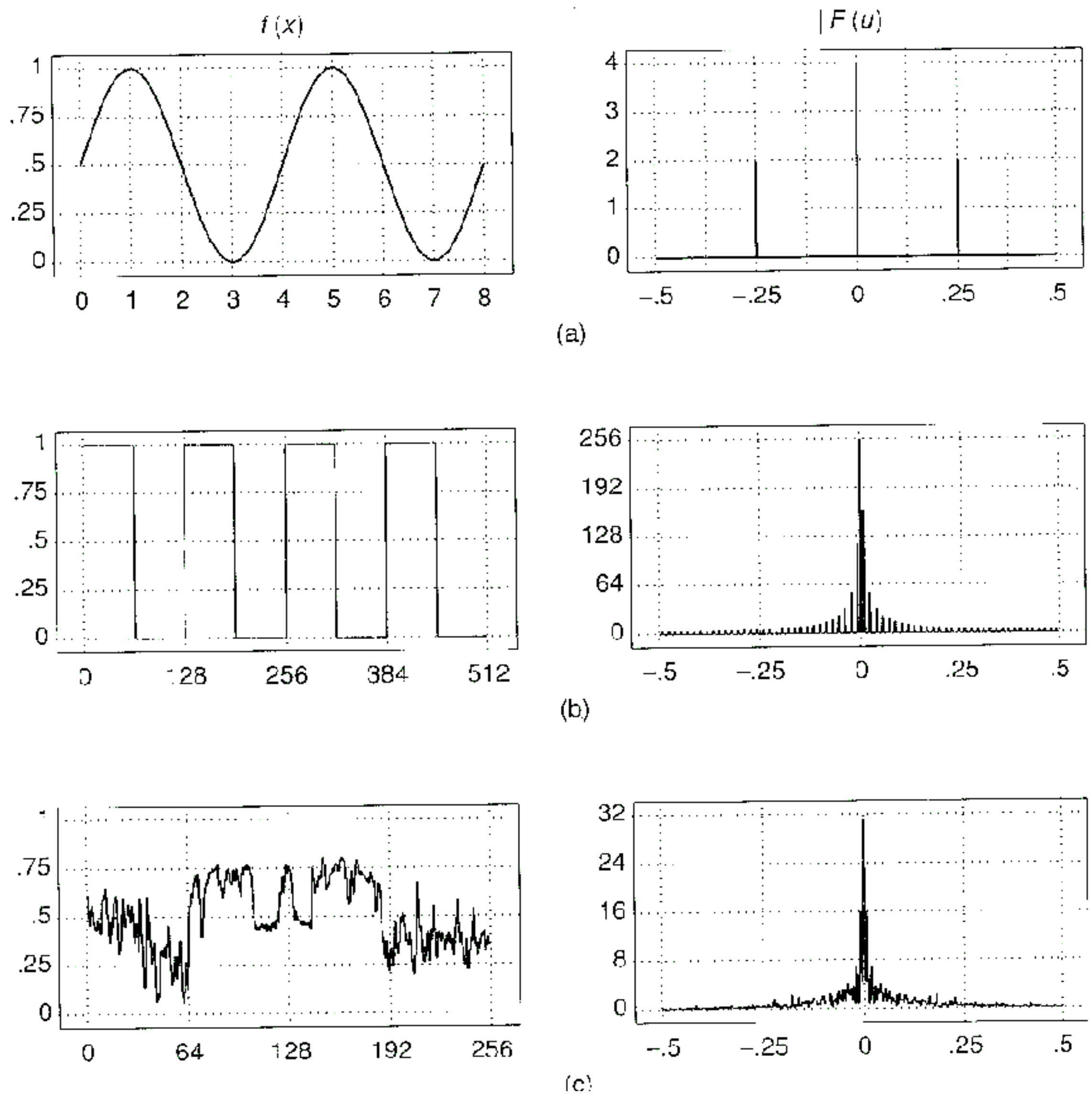
$\Rightarrow$ **Nyquist frequency**

- Sampling rate N_i
- 2 samples / period $\Leftrightarrow$ 0.5 cycles per pixel

$\Rightarrow k \leq N / 2$

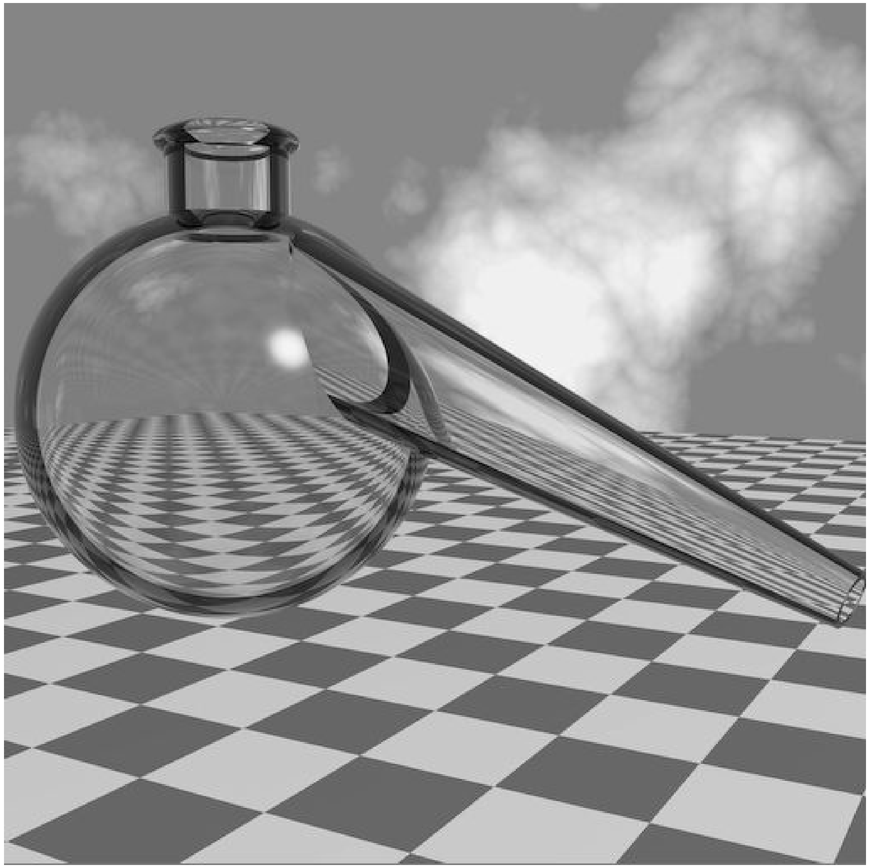
Spatial vs. Frequency Domain

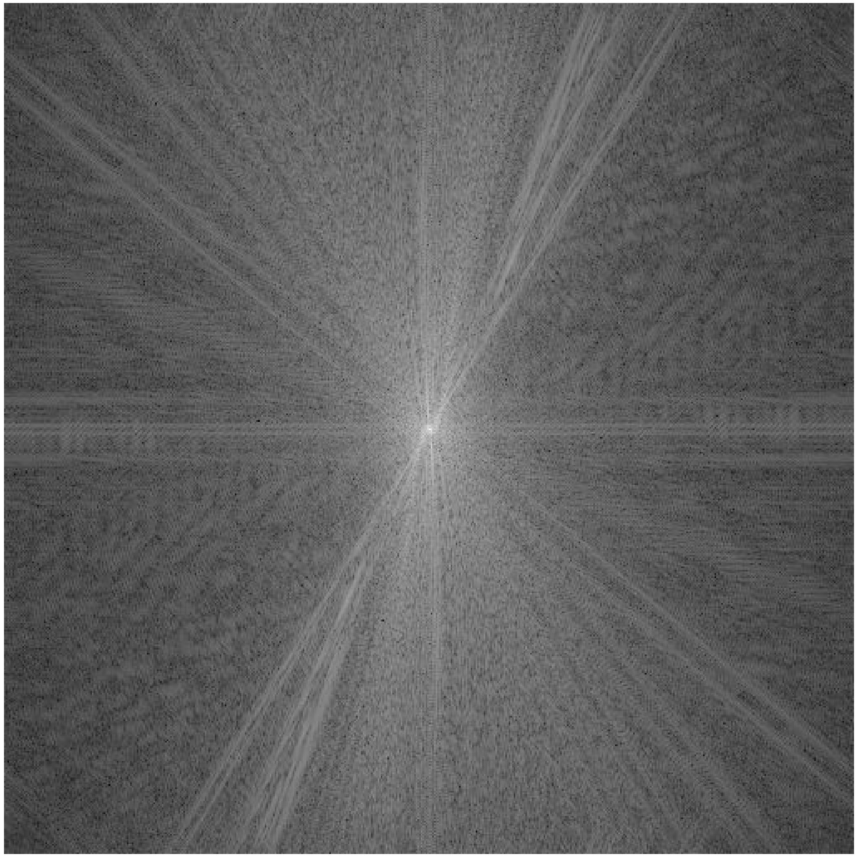
- **Examples** (pixel vs cycles per pixel)
  - Sine wave with positive offset
  - Square wave
  - Scanline of an image



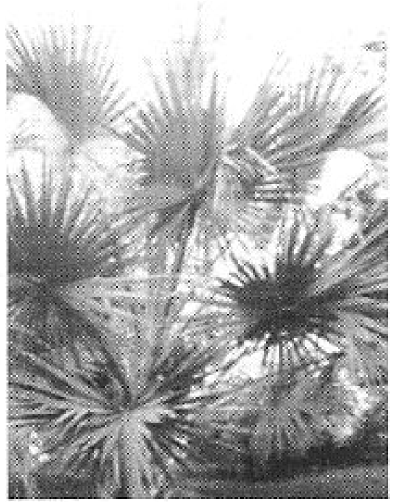
2D Fourier Transform

- 2 separate 1D Fourier transformations along x- and y-direction
- Discontinuities: orthogonal direction in Fourier domain !

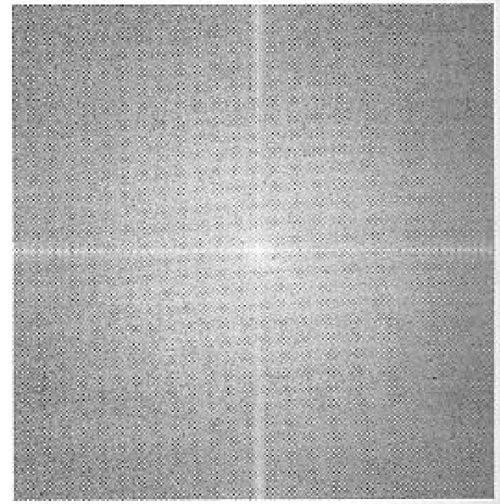




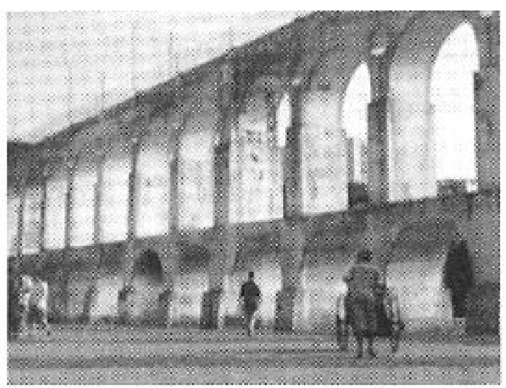
2D Fourier Transforms



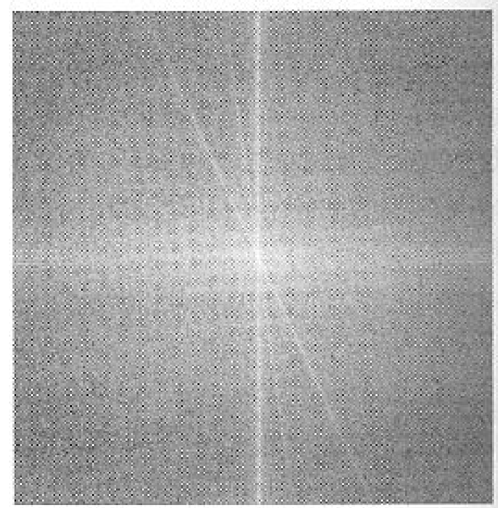
(a) Bush



Fourier transform $|F(u, v)|$



(b) Arcos da Lapa
(Rio de Janeiro)



Fourier transform $|F(u, v)|$

Spatial vs. Frequency Domain

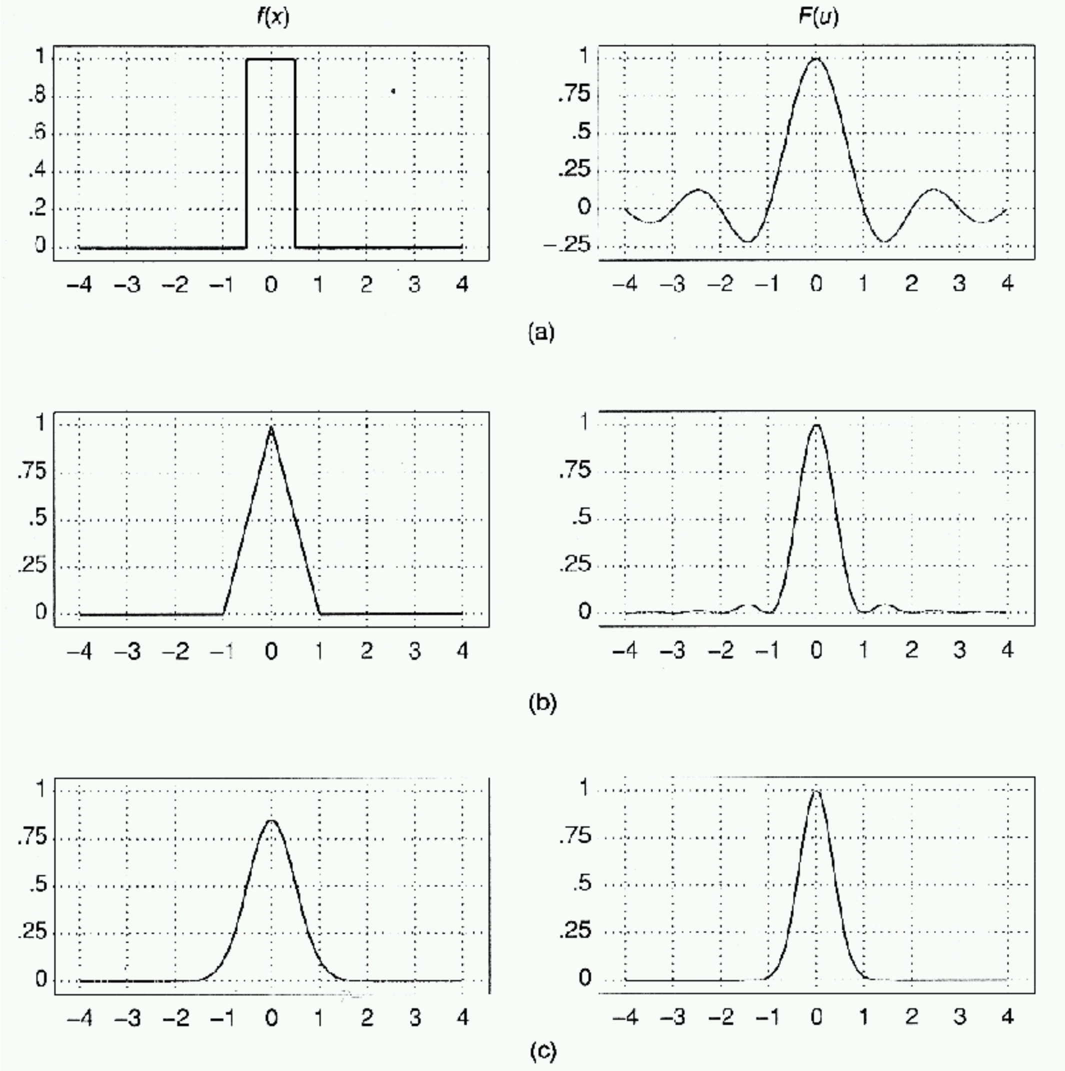
- **Important basis functions**
  - Box ←→ sinc

$$\text{sinc}(x) = \frac{\sin(x\pi)}{x\pi}$$

$$\text{sinc}(0) = 1$$

$$\int \text{sinc}(x)dx = 1$$

    - Wide box → small sinc
    - Negative values
    - Infinite support
  - Triangle ←→ sinc^2
  - Gauss ←→ Gauss

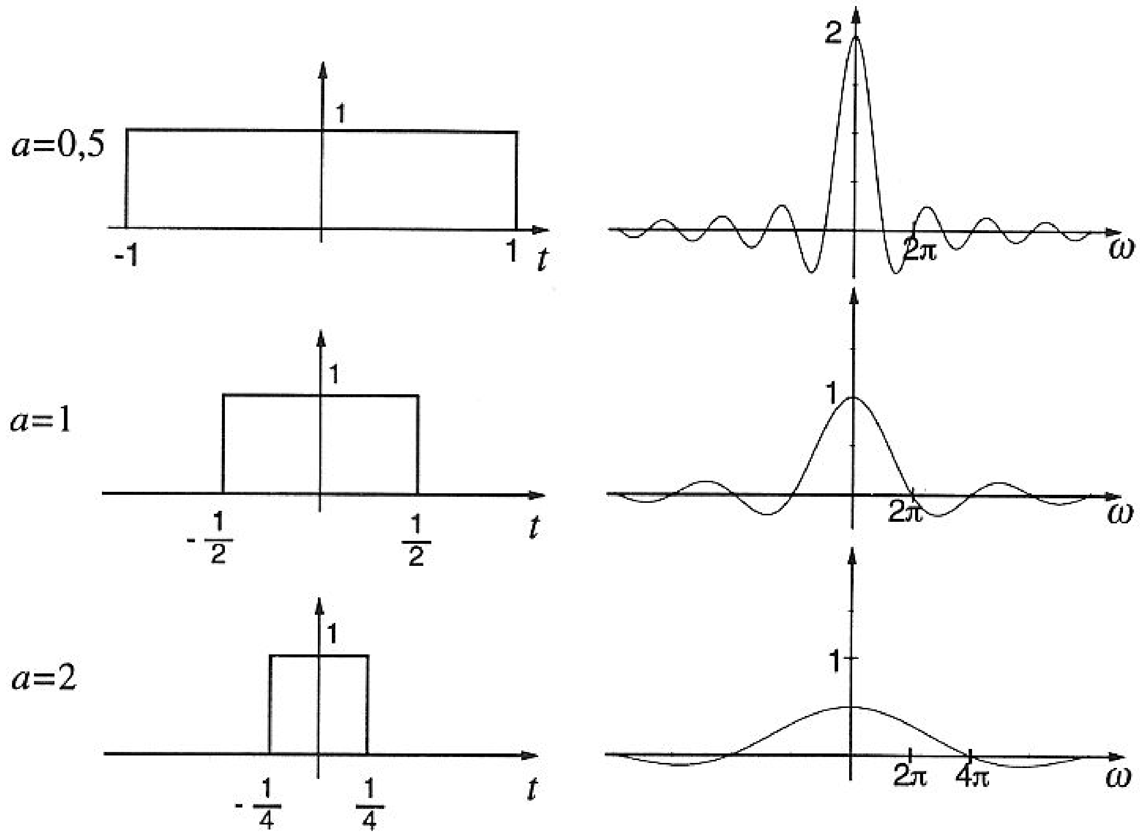


Spatial vs. Frequency Domain

- **Transform behavior**
- **Example: box function**

$$\mathrm{rect}(at) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{|a|}\,\mathrm{si}\left(\frac{\omega}{2a}\right).$$

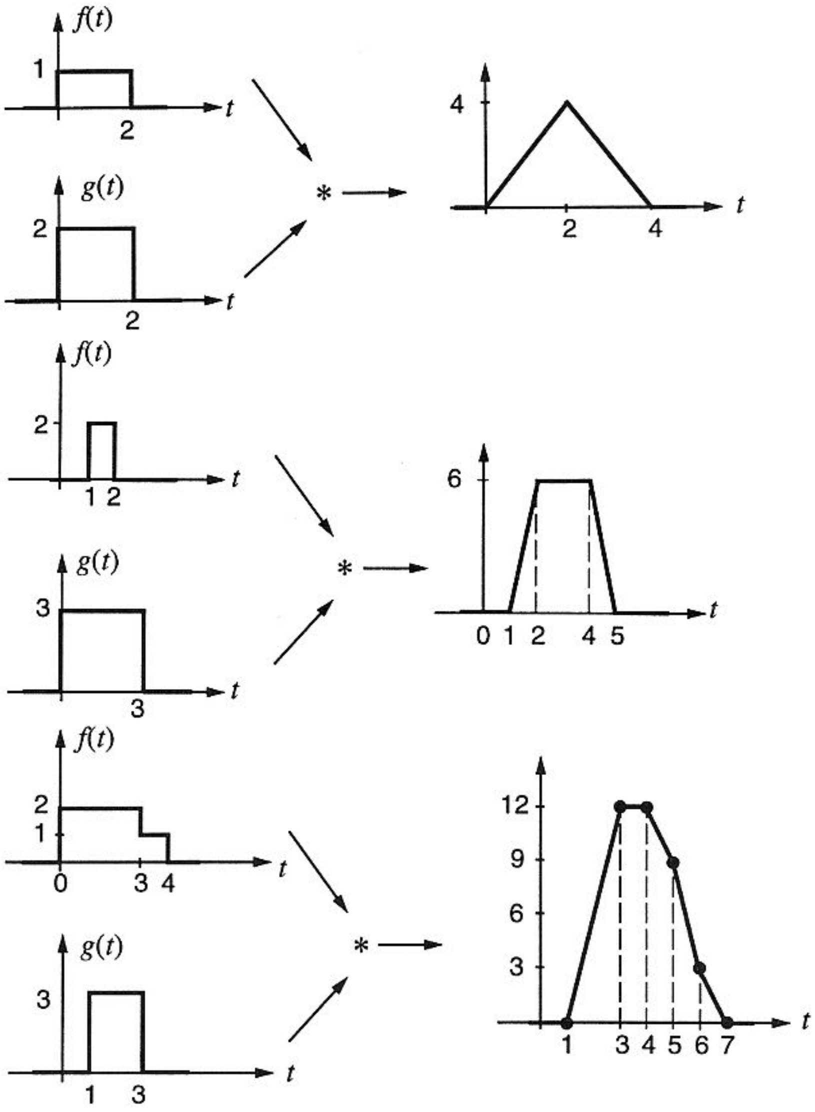
  - Fourier transform: sinc
  - Wide box: narrow sinc
  - Narrow box: wide sinc



Convolution

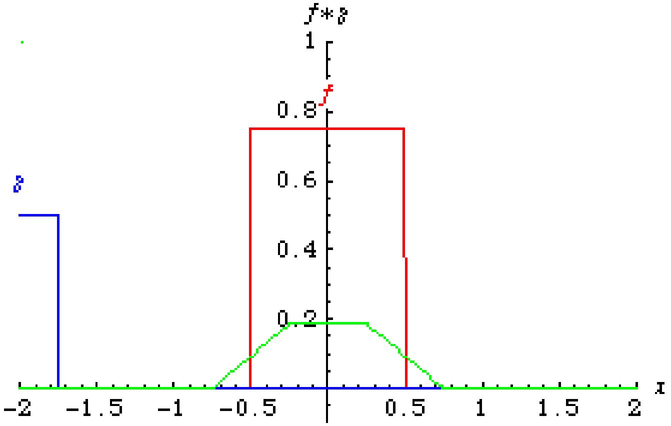
$$f(x) \otimes g(x) = \int_{-\infty}^{\infty} f(\tau)g(x-\tau)d\tau$$

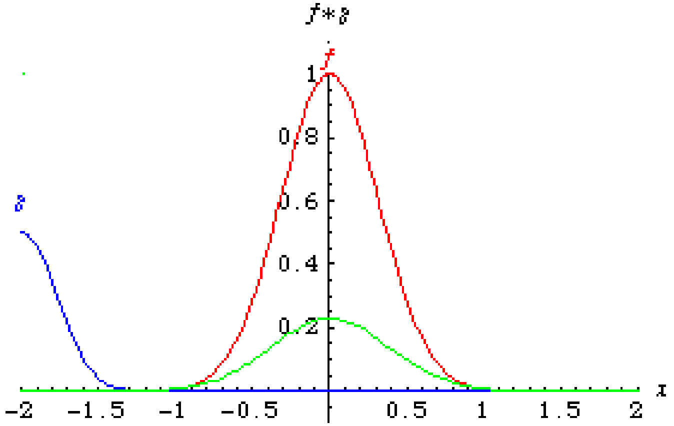
- **Two functions *f, g***
- **Shift one function against the other by *x***
- **Multiply function values**
- **Integrate overlapping region**
- **Numerical convolution: Expensive operation**
  - For each *x:* integrate over non-zero domain



Convolution

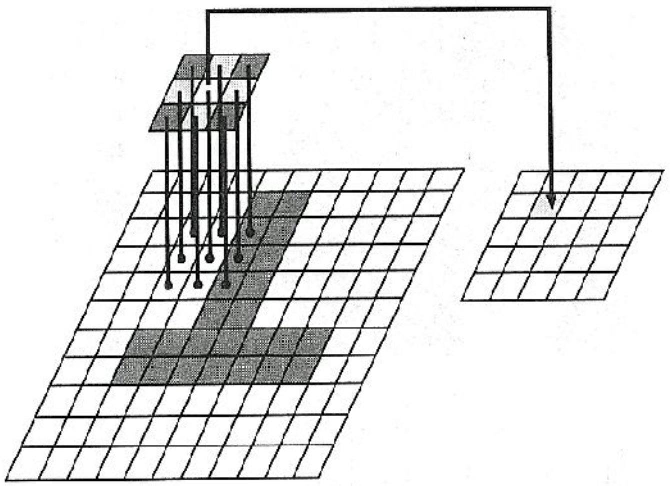
- **Examples**
  - Box functions
  - Gauss functions

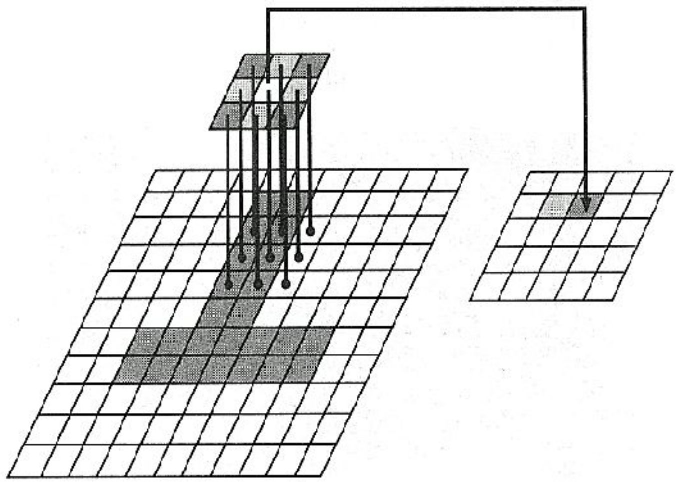




Convolution and Filtering

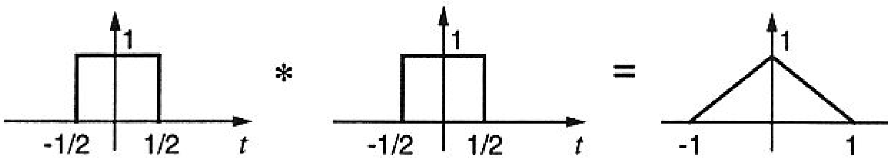
- **Technical Realization**
  - In image domain
  - Pixel mask with weights
  - OpenGL: Convolution extension
- **Problems (e.g. sinc)**
  - Large filter support
    - Large mask
    - A lot of computation
  - Negative weights
    - Negative light?





Convolution Theorem

- **Convolution in image domain: multiplication in Fourier domain**
- **Convolution in Fourier domain: multiplication in image domain**
  - Multiplication much cheaper than convolution !

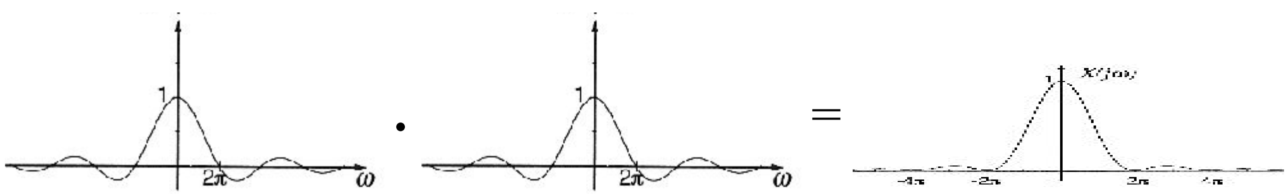


$$x(t) = \begin{cases} t+1 & \text{für } -1 \leq t < 0 \\ -t+1 & \text{für } 0 \leq t < 1 \\ 0 & \text{sonst} \end{cases}$$

$$\mathrm{rect}(t) \ast \mathrm{rect}(t) = x(t)$$

$$\circ\!\!-\!\!\bullet \qquad \circ\!\!-\!\!\bullet \qquad \circ\!\!-\!\!\bullet$$

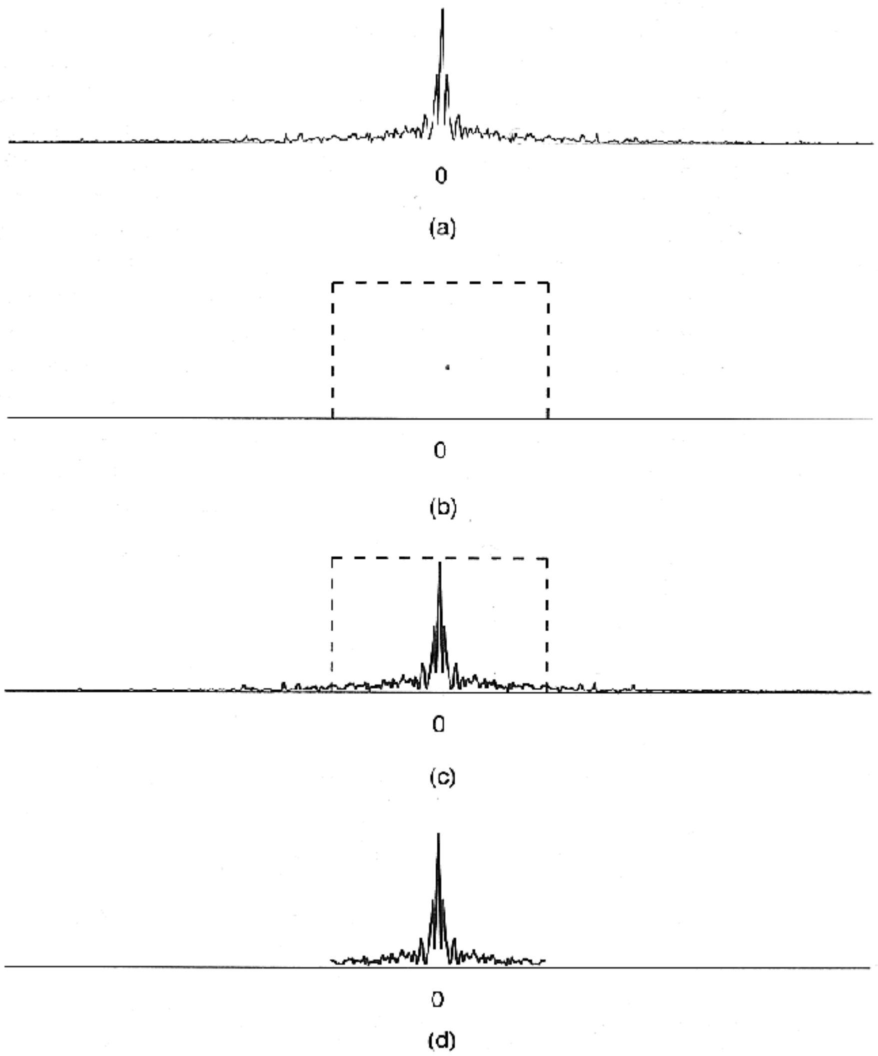
$$\mathrm{si}\left(\frac{\omega}{2}\right) \cdot \mathrm{si}\left(\frac{\omega}{2}\right) = X(j\omega)\,.$$



$$X(j\omega) = \mathrm{si}^2\left(\frac{\omega}{2}\right).$$

Filtering

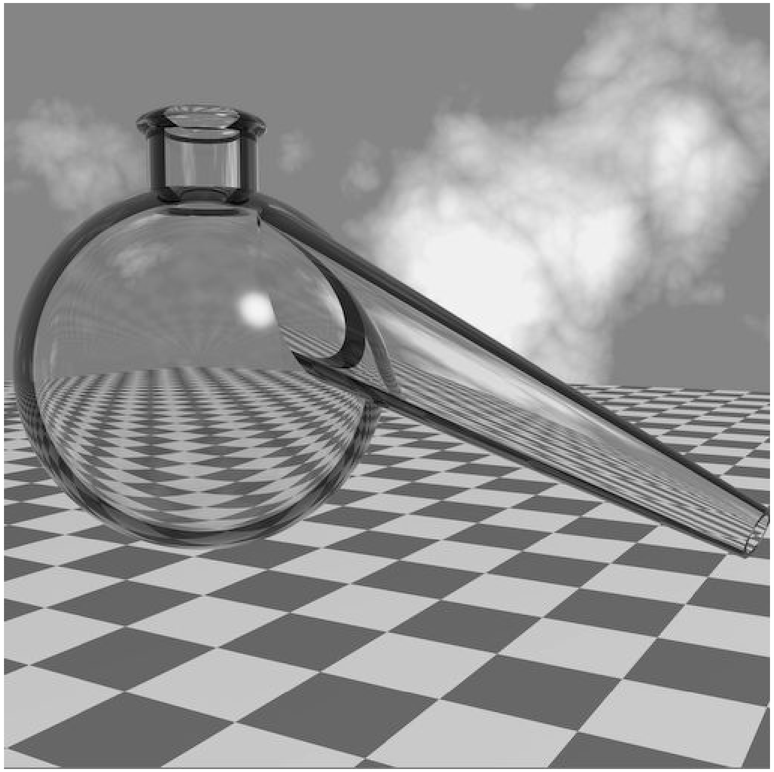
- **Low-pass filtering**
  - Convolution with sinc in spatial domain, or
  - Multiplication with box in frequency domain
- **High-pass filtering**
  - Only high frequencies
- **Band-pass filtering**
  - Only intermediate

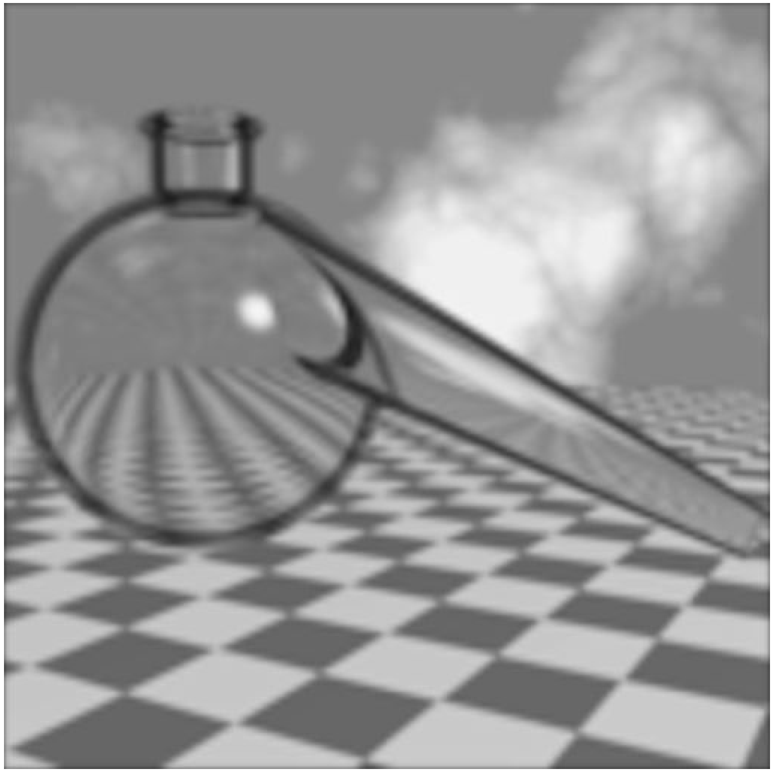


Low-pass filtering in frequency domain: multiplication with box

Low-Pass Filtering

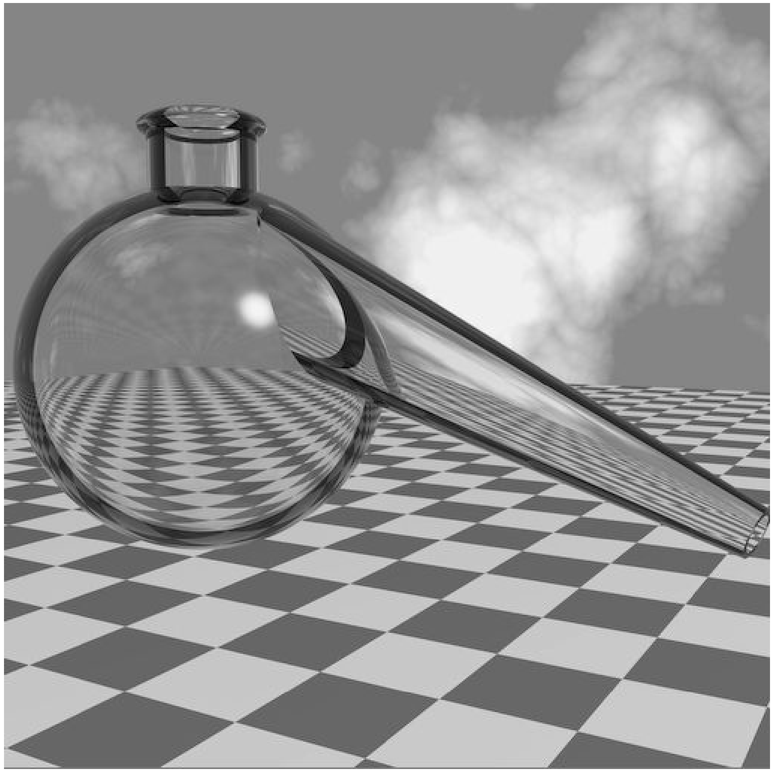
- „Blurring“

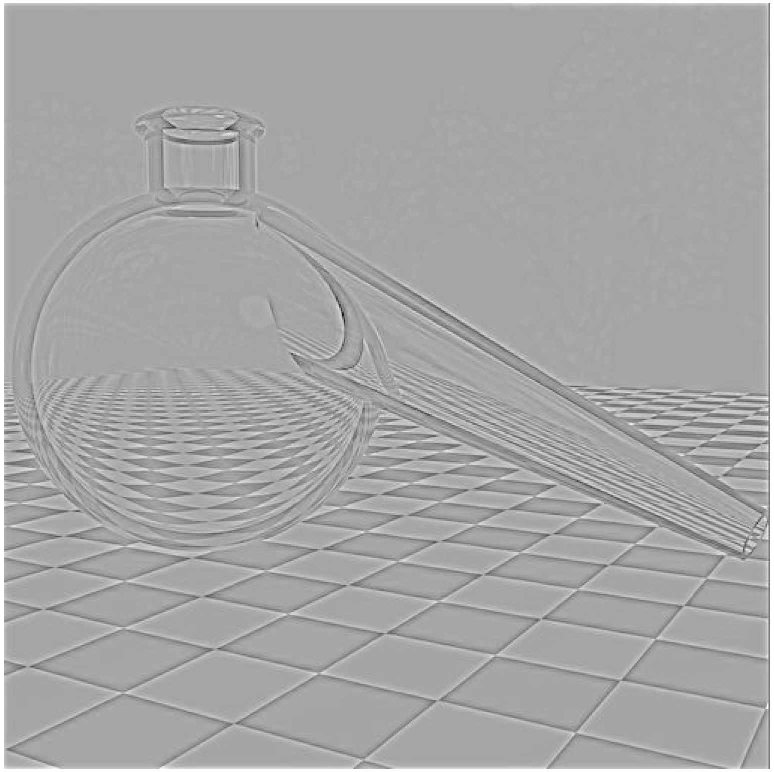




High-Pass Filtering

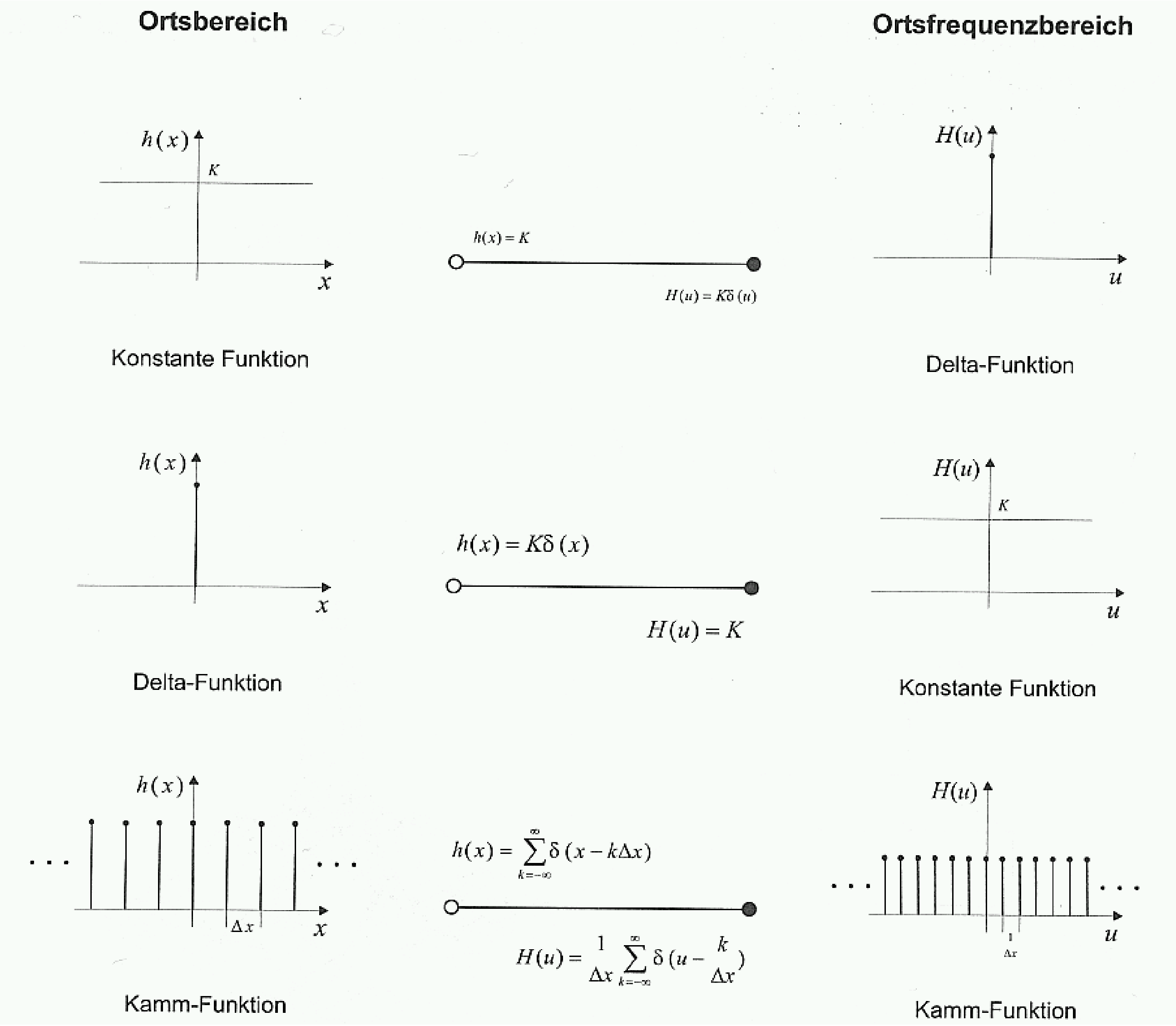
- **Enhances discontinuities in image**
  - Useful for edge detection





Sampling

- **Constant & δ-Function**
  - flash
- **Comb/Shah function**



Sampling

- **Constant & δ-Function**
  - Duality

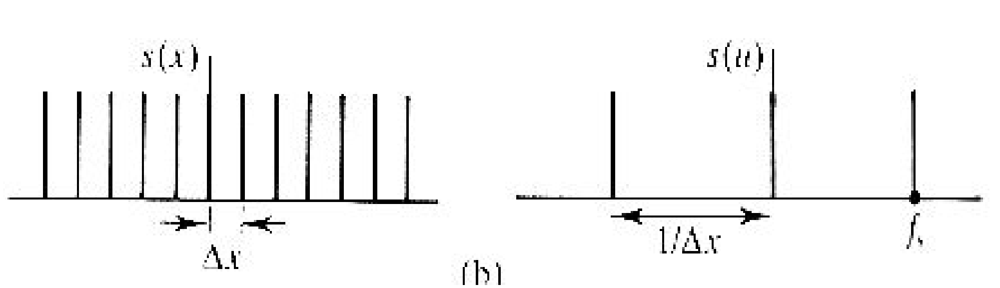
$$f(x) = K$$

$$F(\omega) = K\delta(\omega)$$

  - And vice versa
- **Comb function**
  - Duality: The dual of a comb function is again a comb function
    - Inverse wave length, amplitude scales with inverse wave length

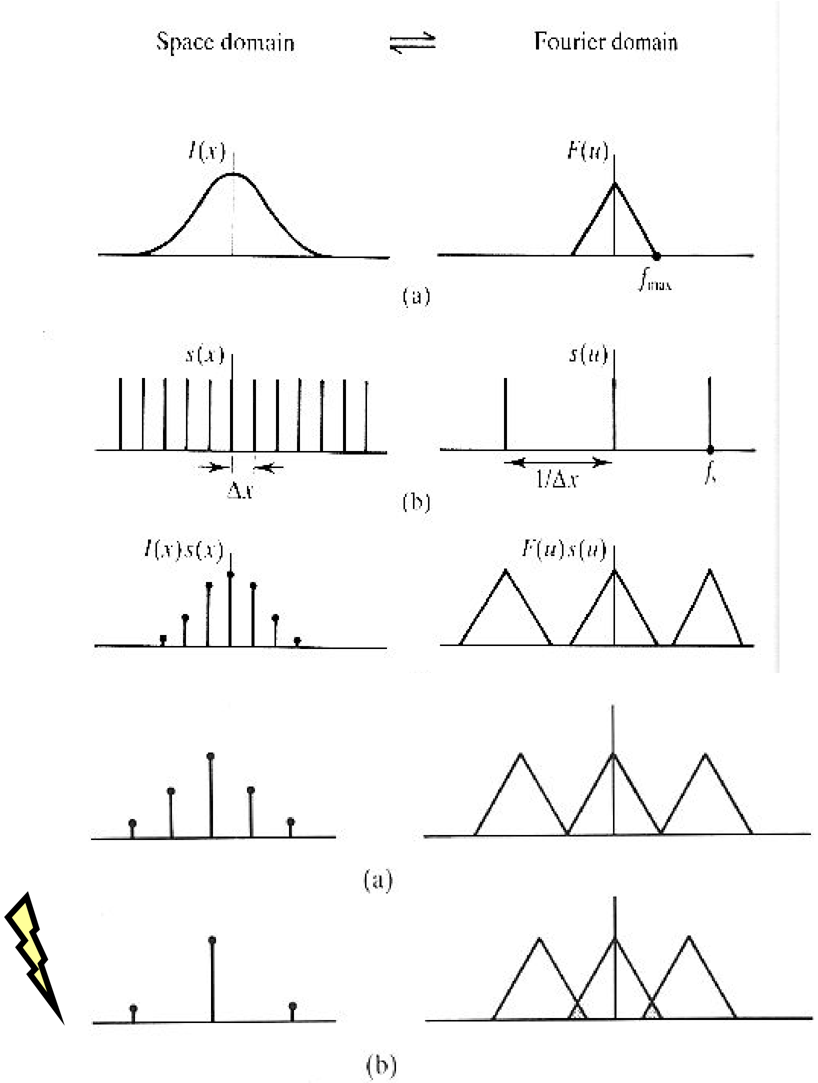
$$f(x) = \sum_{k=-\infty}^{\infty} \delta(x - k\Delta x)$$

$$F(w) = \frac{1}{\Delta x} \sum_{k=-\infty}^{\infty} \delta\left(w - k\frac{1}{\Delta x}\right)$$



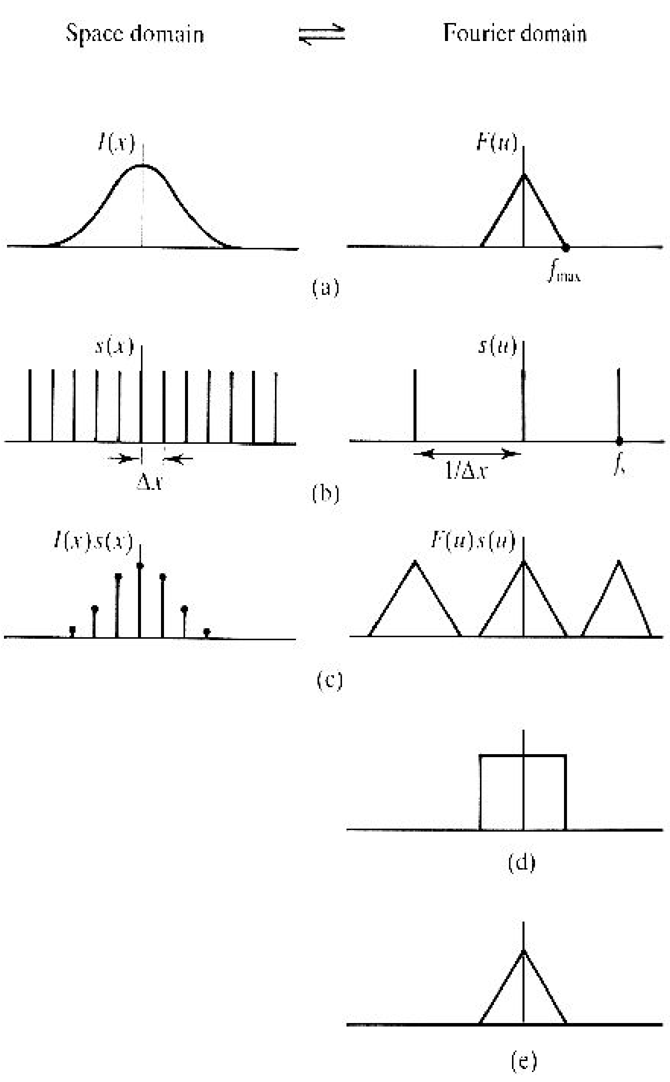
Sampling

- **Continuous function**
  - Band-limited Fourier transform
- **Sampled at discrete points**
  - Multiplication with Comb function in space domain
  - Corresponds to convolution in Fourier domain
- **Frequency bands overlap ?**
  - No: good
  - Yes: bad, ***aliasing***



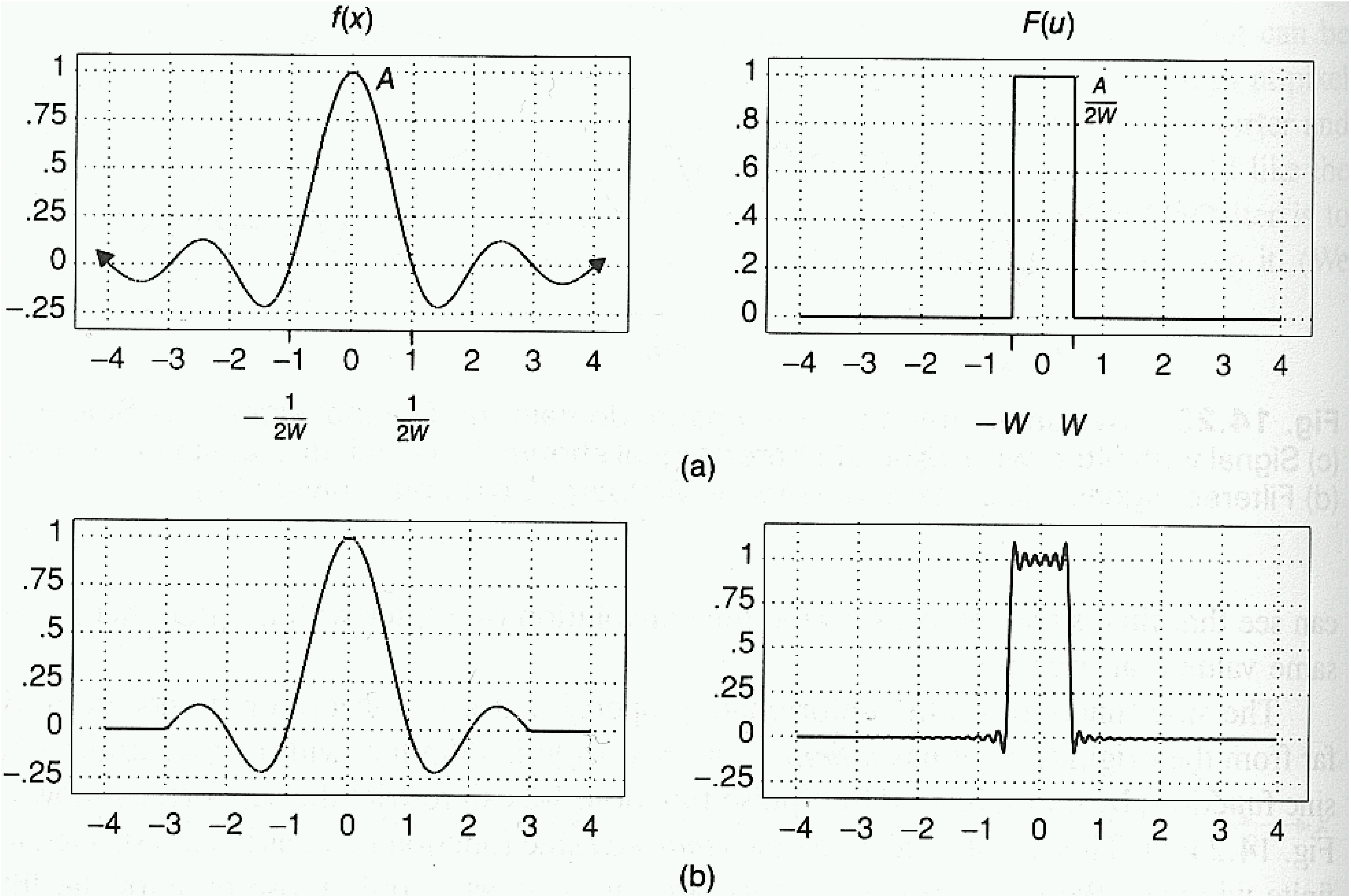
Reconstruction

- **Only original frequency band desired**
- **Filtering**
  - In Fourier domain: multiplication with windowing function around origin
  - In spatial domain: convolution with Fourier transform of windowing function
- **Optimal filtering function**
  - Box function in Fourier domain
  - Corresponds to sinc in space domain
    - Unlimited region of support
  - Spatial domain only allows approximations (limited support)



Reconstruction Filter

- **Cutting off the support is *not* a good solution**



Sampling and Reconstruction

Original function and its band-limited frequency spectrum

Signal sampling:

Mult./conv. with comb

Comb dense enough (sampling≥2*bandlimit)

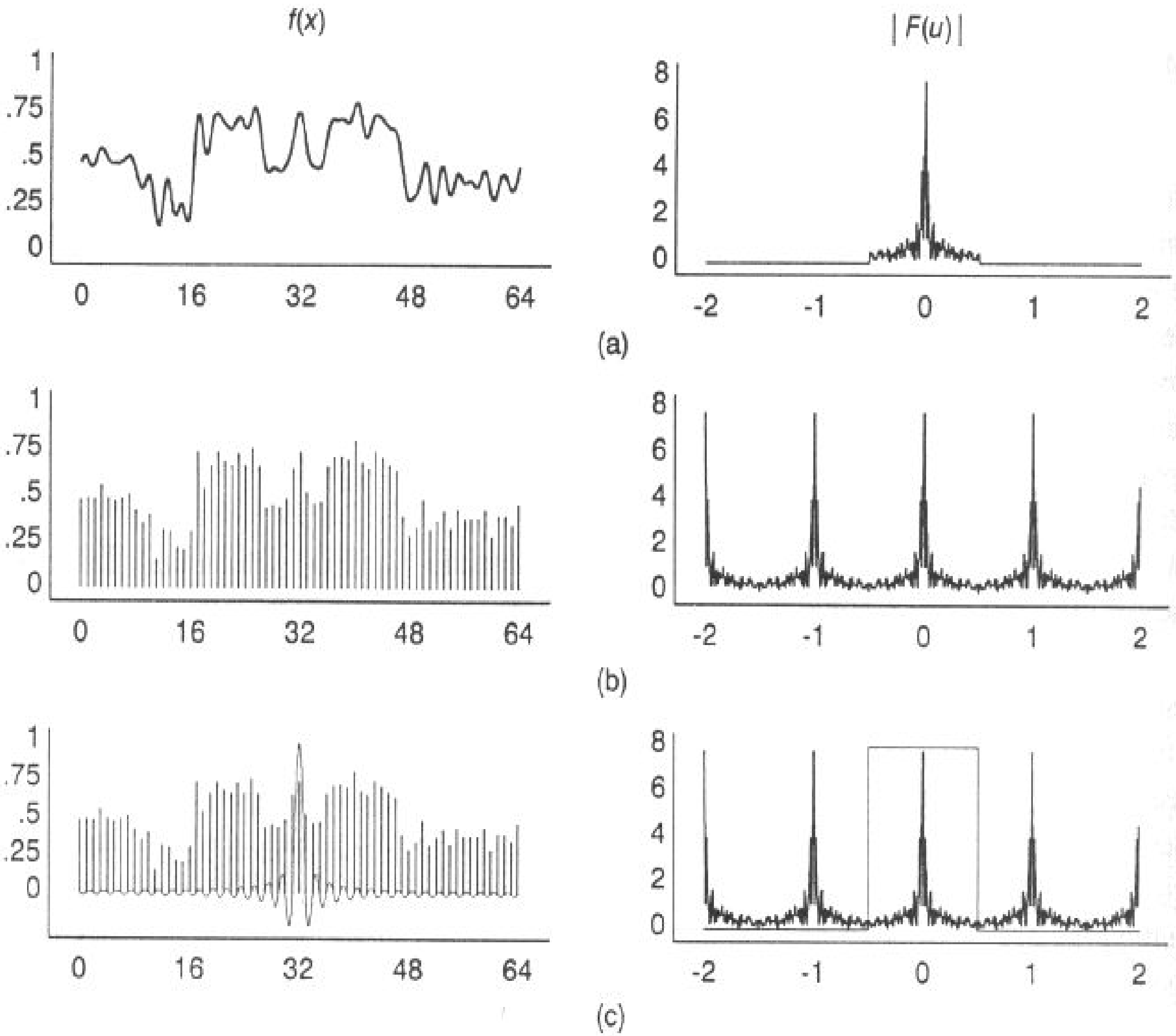
Frequency spectrum is replicated

Bands do not overlap

Correct filtering

Fourier: Box (mult.)
Image: sinc (conv.)

Only one copy



Sampling and Reconstruction

Reconstruction with ideal sinc

Identical signal

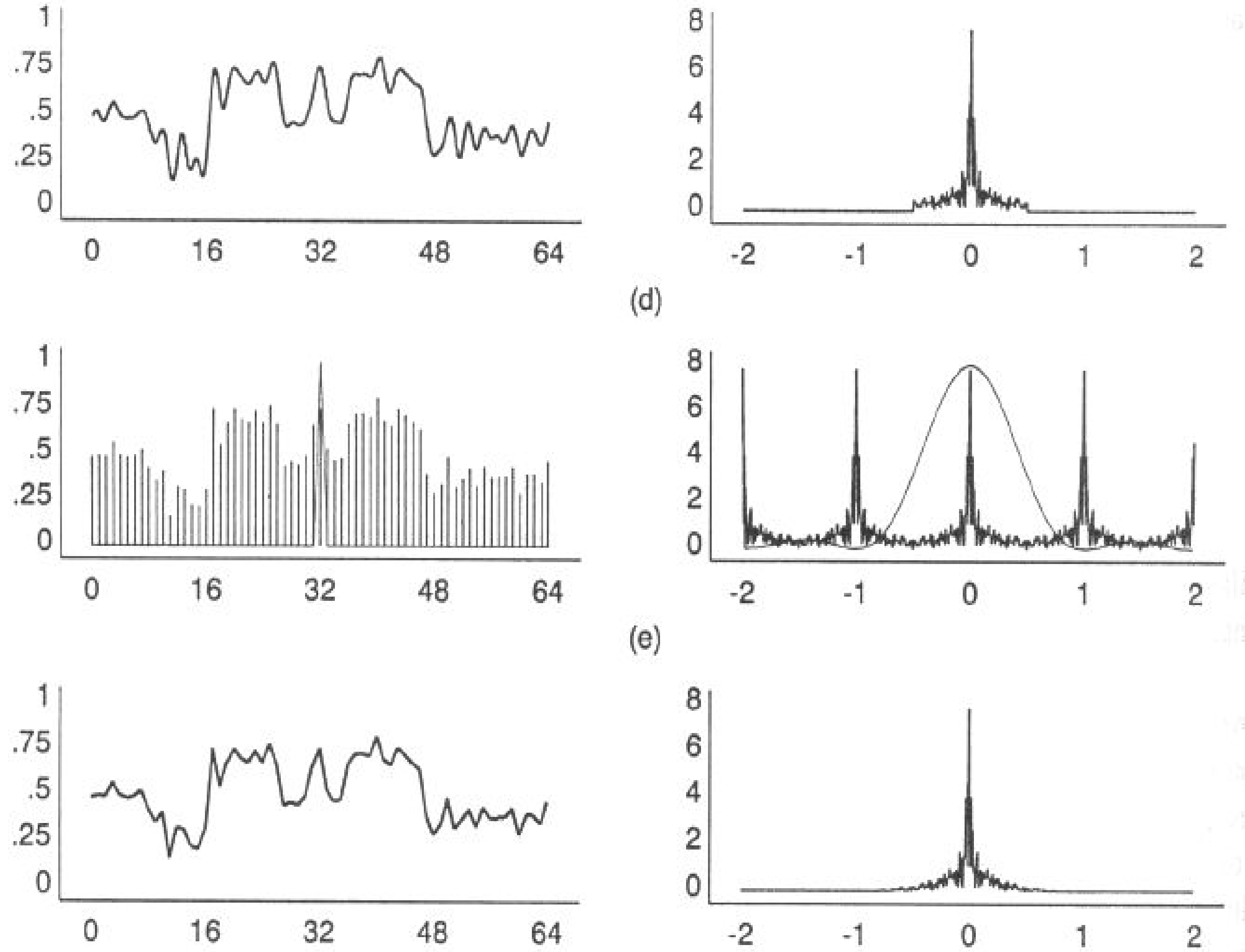
Approximate filtering

Space: tri (conv.)
Fourier: sinc² (mult.)

High frequencies are not ignored

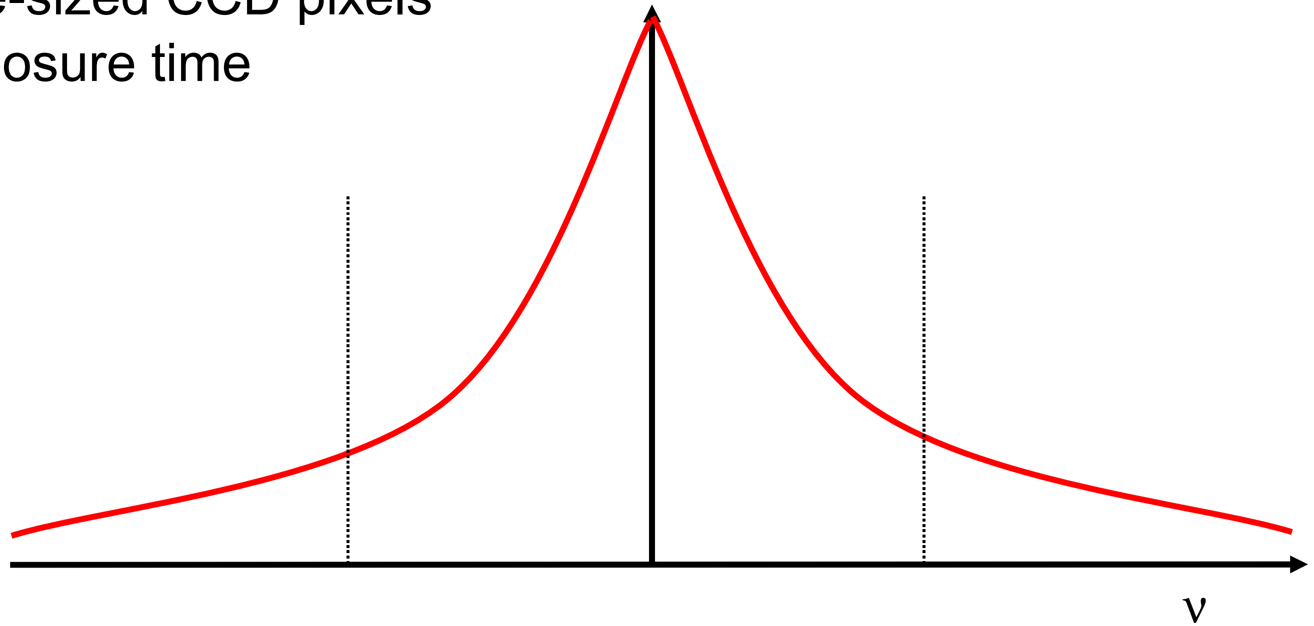
➔ Aliasing

Reconstruction with tri function (= piecewise linear interpolation)



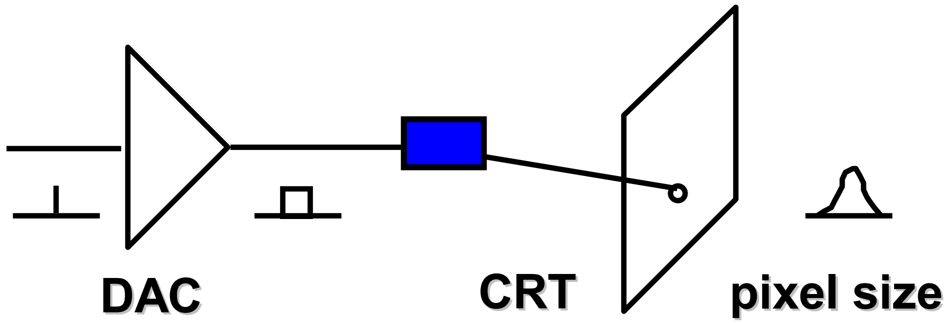
Filtering during Acquisition

- **Band-limit unlimited natural signal**
  - Finite spectral range
  - Makes non-aliased reconstruction possible
  - Example: thermal noise
- **Natural low-pass filtering**
  - E.g., finite-sized CCD pixels
  - Finite exposure time



Filtering during Display (CRT)

- **Reconstruction**
  – Physical output devices generate a continuous signal, even for discrete input, e.g., on a computer monitor



- **Example**
  – DAC (Digital-to-Analog Converter): Sample and hold
    - Capacities and inductivities
  – CRT: Phosphor and light spot
    - Afterglow on screen

Sampling with Low Frequency

Original function

Sampling below Nyquist:

Comb spaced to far (sampling<2*bandlimit)

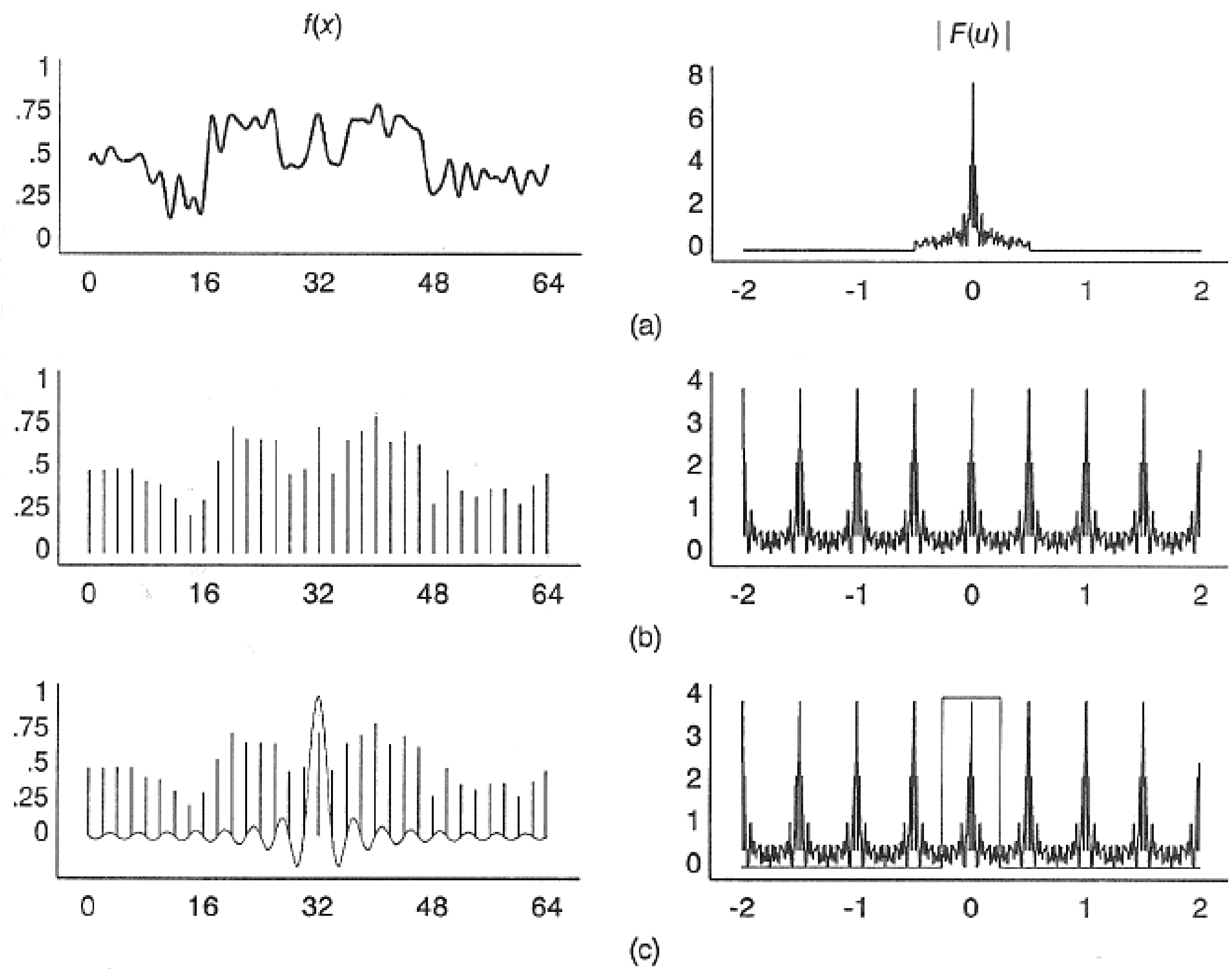
Frequency bands overlap

Correct filtering

Image: sinc (conv.)

Fourier: hat (mult.)

Band overlap in frequency domain cannot be corrected - aliasing



Sampling with Low Frequency

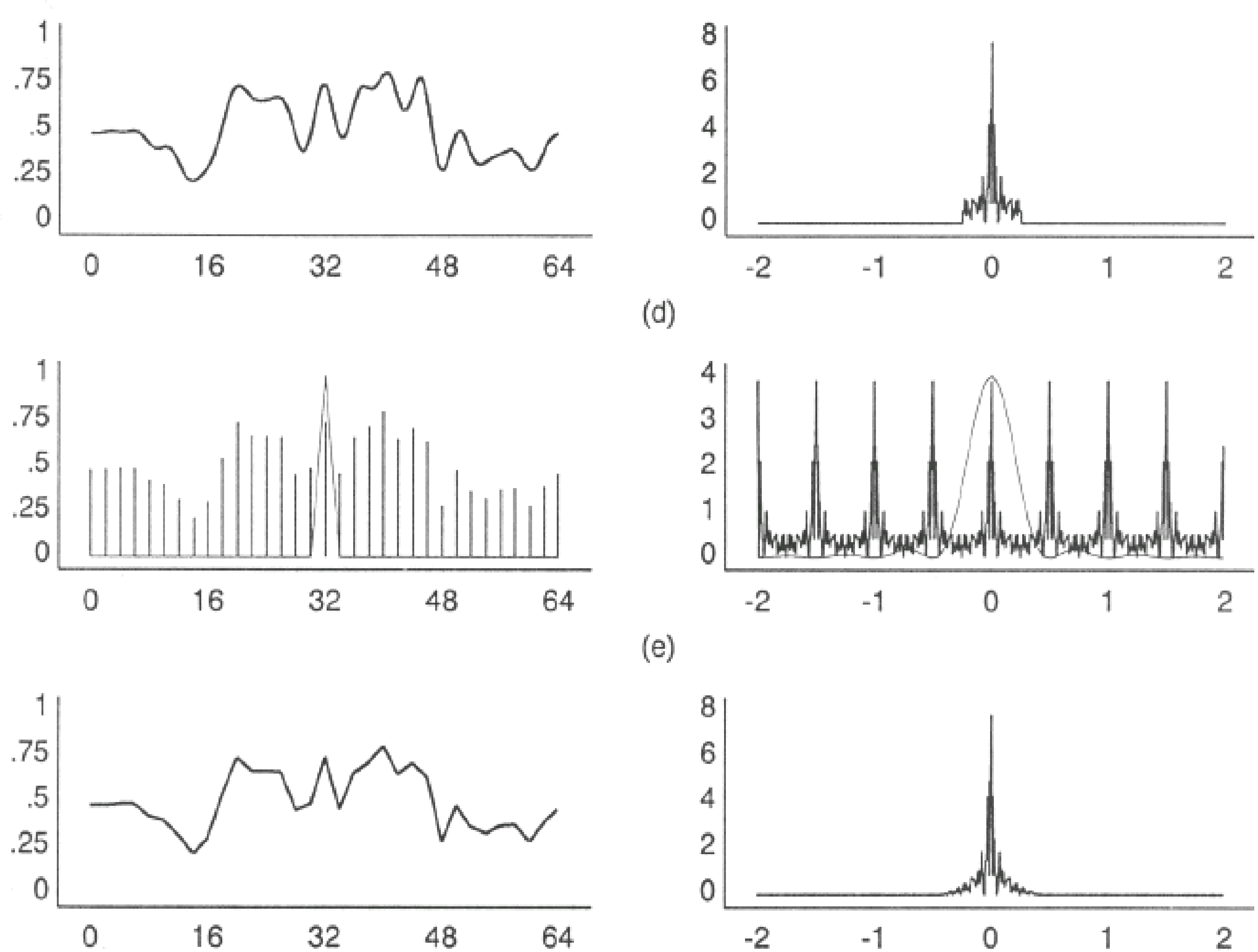
Reconstruction with ideal sinc

Reconstruction fails (frequency components wrong due to aliasing !)

Fitlering with sinc² function

Reconstruction with tri function (= piecewise linear interpolation)

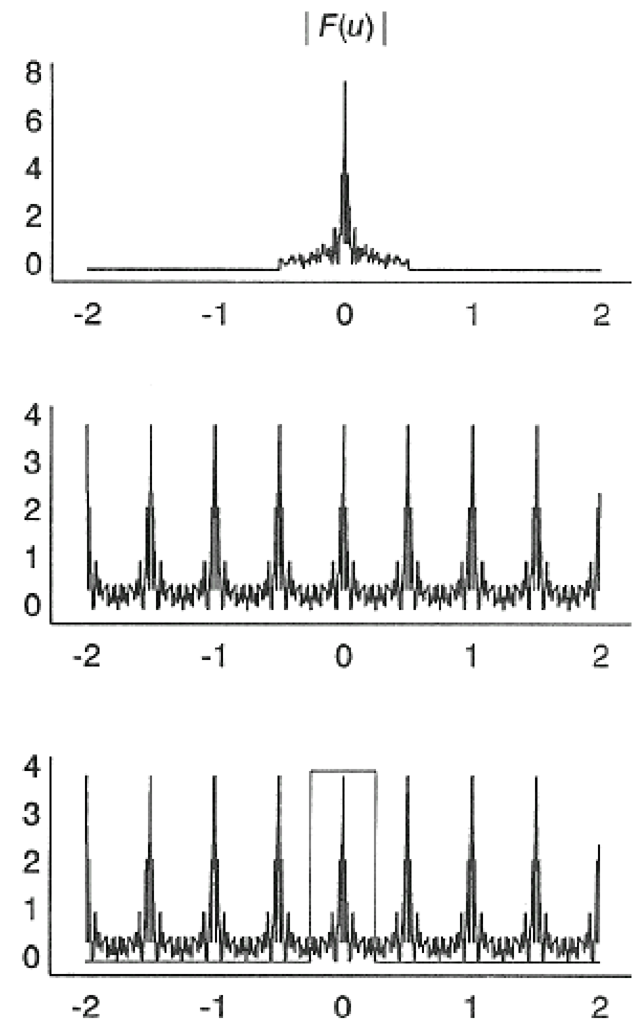
Even worse reconstruction



Aliasing

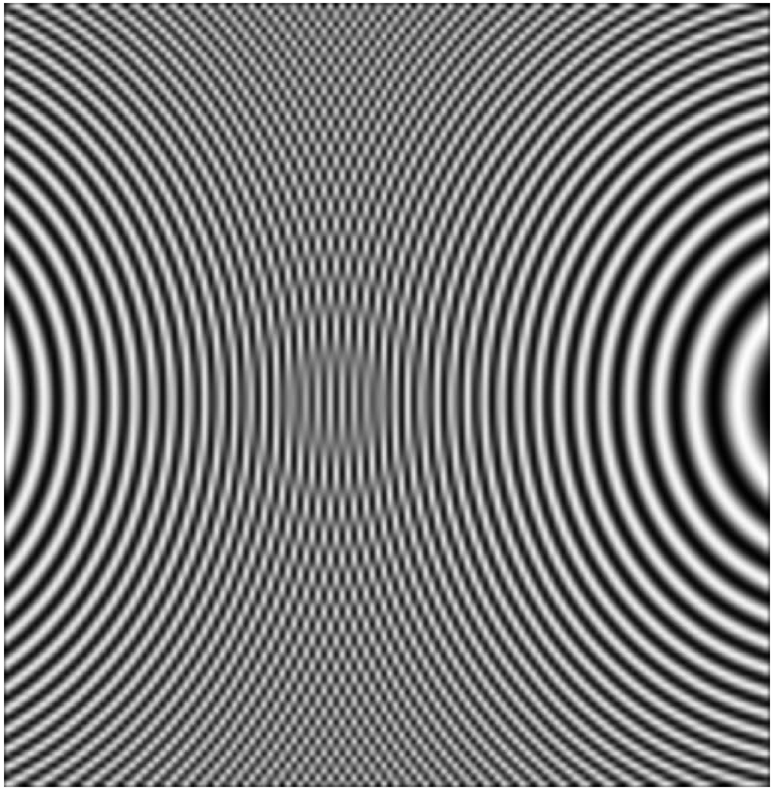
- **Overlap between replicated copies in frequency spectrum**

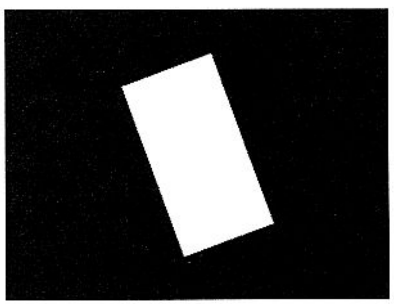
  High frequency components from the replicated copies are treated like low frequencies during the reconstruction process



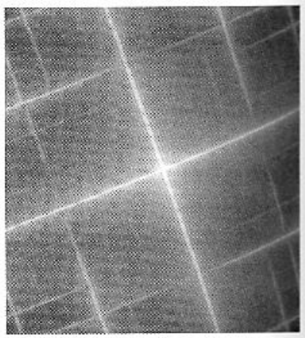
Aliasing Artifacts

- **Moiré patterns**
- **Aliasing**
- **Jaggies**

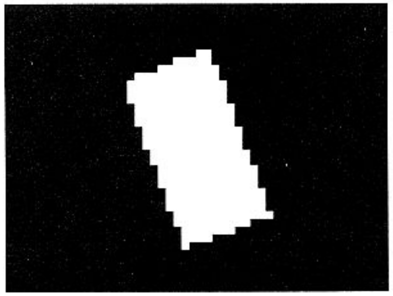




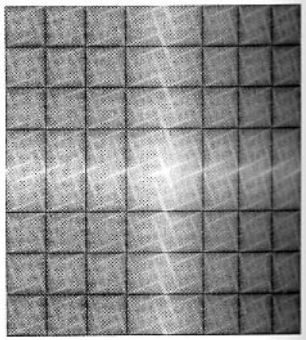
(a) Simulation of a perfect line



(b) Fourier transform of (a)



(c) Simulation of a jagged line



(d) Fourier transform of (c)

Wrap-Up

- **Fourier transformation**
  - Equivalent representation of transformed signal
  - Spectral analysis: shows signal's frequency components
- **Convolution**
  - Filtering
- **Sampling**
  - Multiplication with comb function
  - Only at discrete points: no integration over signal
  - Frequency spectrum replicated
  - Replication distance: sampling rate
- **Aliasing**
  - Replicated spectra overlap
  - Cannot be separated by filtering anymore
  - Erroneous frequency amplitudes – wrong function !