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Computer Graphics I
Assignment 9

Submission deadline for the exercises: Thursday, 25th January 2007

9.1 Homogenous Coordinates (5 + 10 + 15 Points)

a) Show that multiplying the homogenous point (x, y, z, w) with an arbitrary scalar $\alpha \neq 0$ yields an equivalent homogenous point again.

b) Show that the component wise addition of three homogenous points $(a_0, b_0, c_0, 1)$, $(a_1, b_1, c_1, 1)$, and $(a_2, b_2, c_2, 1)$ yields the center between that points.

c) Find an addition rule in homogenous space that is equivalent to standard addition after performing the projection.

9.2 Perspective Projection (30 Points)

In the lecture a perspective projection P_{persp} is given that maps the viewing frustum to the regular box $[-1, 1]^3$. In this exercise you have to derive a formula for a similar projection P' from scratch, that maps the near-plane to the $z = 0$ plane and the far-plane to the $z = 1$ plane. The viewing direction is the positive z direction, such that the viewing frustum has an x extension from $-\frac{w}{2}$ to $\frac{w}{2}$ and y extension from $-\frac{h}{2}$ to $\frac{h}{2}$ in the near plane. Find the corresponding transformation P':

$$P' = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

9.3 Perspective Projection (15 + 15 + 10 Points)

a) Compute the point where two arbitrary parallel lines seem to intersect after being projected by the perspective projection P. For which parallel lines does no such intersection point exist?

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}$$

b) Compute the center of projection of the following perspective projection Q.

$$Q = \begin{pmatrix} \frac{3}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} \\ -1 & 0 & 0 & -1 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{5}{4} \end{pmatrix}$$

c) Compute the projection plane of projection Q'

$$Q' = \begin{pmatrix} 1 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{5}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}$$

9.4 Homogenous Lines in 2D* (20 Points)

Prove that the cross product between two homogenous points $p = (p_x, p_y, p_w)$ and $q = (q_x, q_y, q_w)$ yields the homogenous coordinates of the connecting line.

Solutions

9.1 Homogenous Coordinates

a) Both (x, y, z, w) and $\alpha \cdot (x, y, z, w) = (\alpha x, \alpha y, \alpha z, \alpha w)$ are representatives of the same point $\left(\frac{\alpha x}{\alpha w}, \frac{\alpha y}{\alpha w}, \frac{\alpha z}{\alpha w}\right) \overset{\alpha \neq 0}{=} \left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)$ and thus two equivalent homogenous coordinates.

b) The addition of the three vectors gives:

$$\begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ 1 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ 1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_0 + a_1 + a_2 \\ b_0 + b_1 + b_2 \\ c_0 + c_1 + c_2 \\ 3 \end{pmatrix} \overset{\texttt{project}}{=} \begin{pmatrix} \frac{a_0+a_1+a_2}{3} \\ \frac{b_0+b_1+b_2}{3} \\ \frac{c_0+c_1+c_2}{3} \end{pmatrix}$$

Thus the addition of the three homogenous vectors is a representation of the center of mass of them.

c) To find this rule we first project two homogenous vectors and then reproject again:

$$\begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ w_0 \end{pmatrix} \oplus \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ w_1 \end{pmatrix} \overset{\texttt{project}}{=} \begin{pmatrix} \frac{a_0}{w_0} \\ \frac{b_0}{w_0} \\ \frac{c_0}{w_0} \end{pmatrix} + \begin{pmatrix} \frac{a_1}{w_1} \\ \frac{b_1}{w_1} \\ \frac{c_1}{w_1} \end{pmatrix} = \begin{pmatrix} \frac{a_0}{w_0} + \frac{a_1}{w_1} \\ \frac{b_0}{w_0} + \frac{b_1}{w_1} \\ \frac{c_0}{w_0} + \frac{c_1}{w_1} \end{pmatrix} = \begin{pmatrix} \frac{a_0 w_1 + a_1 w_0}{w_0 w_1} \\ \frac{b_0 w_1 + b_1 w_0}{w_0 w_1} \\ \frac{c_0 w_1 + c_1 w_0}{w_0 w_1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a_0 w_1}{w_0 w_1} + \frac{a_1 w_0}{w_0 w_1} \\ \frac{b_0 w_1}{w_0 w_1} + \frac{b_1 w_0}{w_0 w_1} \\ \frac{c_0 w_1}{w_0 w_1} + \frac{c_1 w_0}{w_0 w_1} \end{pmatrix} = \begin{pmatrix} \frac{a_0 w_1}{w_0 w_1} \\ \frac{b_0 w_1}{w_0 w_1} \\ \frac{c_0 w_1}{w_0 w_1} \end{pmatrix} + \begin{pmatrix} \frac{a_1 w_0}{w_0 w_1} \\ \frac{b_1 w_0}{w_0 w_1} \\ \frac{c_1 w_0}{w_0 w_1} \end{pmatrix}$$

$$\overset{\texttt{re-project}}{=} w_1 \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ w_0 \end{pmatrix} + w_0 \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ w_1 \end{pmatrix}$$

9.2 Perspective Projection

We get a system of equations by mapping two appropriate edge points of the viewing frustum:

$$P' \cdot \left(-\frac{w}{2}, -\frac{h}{2}, near, 1\right)^T = (-1, -1, 0, 1)^T$$

$$P' \cdot \left(\frac{w}{2} \cdot \frac{far}{near}, \frac{h}{2} \cdot \frac{far}{near}, far, 1\right)^T = (1, 1, 1, 1)^T$$

This gives us:

$$\begin{pmatrix} -a\frac{w}{2} \\ -b\frac{h}{2} \\ c\ near + d \\ near \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} a\frac{w}{2} \cdot \frac{far}{near} \\ b\frac{h}{2} \cdot \frac{far}{near} \\ c\ far + d \\ far \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Performing the projection gives these equations:

$$-a\frac{w}{2} = -near \tag{1}$$

$$-b\frac{h}{2} = -near \tag{2}$$

$$c\ near + d = 0 \tag{3}$$

$$a\frac{w}{2}\frac{far}{near} = far \tag{4}$$

$$
\begin{aligned}
b\frac{w}{2}\frac{far}{near} &= far & (5)\\
c\, far + d &= far & (6)\\
& & (7)
\end{aligned}
$$

Equation (1) and (4) as well as (2) and (5) are linear dependent. Equation (1) gives $a = \frac{2\,near}{w}$ and (2) gives $b = \frac{2\,near}{h}$. Using the two equations (3) and (6) we can compute c and d as:

$$(6) - (3) \Leftrightarrow c(far - near) = far \Leftrightarrow c = \frac{far}{far - near}$$

The value d can now simply be computed by inserting c into equation (3), which yields:

$$d = -\frac{near far}{far - near}$$

The solution is:

$$P' = \begin{pmatrix} \frac{2\,near}{w} & 0 & 0 & 0 \\ 0 & \frac{2\,near}{h} & 0 & 0 \\ 0 & 0 & \frac{far}{far-near} & -\frac{nearfar}{far-near} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

9.3 Perspective Projection

a) We consider the line $h(\lambda) = \lambda v + p = \lambda(v_x, v_y, v_z, 0)^T + (p_x, p_y, p_z, 1)^T$. If we apply the transformation P we get:

$$P(h(\lambda)) = \lambda \begin{pmatrix} v_x \\ v_y \\ 0 \\ \frac{v_z}{2} \end{pmatrix} + \begin{pmatrix} p_x \\ p_y \\ 0 \\ \frac{p_z}{2} + 1 \end{pmatrix}$$

As multiplication with a scalar does not affect the homogenous vector we can multiply the result with $1/\lambda$ and get:

$$P(h(\lambda)) = \begin{pmatrix} v_x \\ v_y \\ 0 \\ \frac{v_z}{2} \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} p_x \\ p_y \\ 0 \\ \frac{p_z}{2} + 1 \end{pmatrix}$$

For $\lambda \to +\infty$ the right term goes to zero and we get the point $p_v = (v_x, v_y, 0, \frac{v_z}{2})^T$ which is independent of p and thus it is the homogenous vanishing point for all lines with direction v. After projection we get:

$$p_v' = \left(\frac{2v_x}{v_z}, \frac{2v_y}{v_z}, 0, 1\right)^T$$

This give us the vanishing point $\left(\frac{2v_x}{v_z}, \frac{2v_y}{v_z}\right)^T$ after the projection. There exists no such point if the lines are parallel to the x-y plane, this is if $v_z = 0$ as then the division is not defined.

b) The projection center is mapped to the homogenous point $(0,0,0,0)^T$ by the perspective projection. This yields a linear system of equations:

$$\begin{pmatrix} \frac{3}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} \\ -1 & 0 & 0 & -1 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{5}{4} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{3}{4}x - \frac{1}{4}z - \frac{1}{4} = 0 \Rightarrow 3x - z - 1 = 0 \tag{8}$$

$$\frac{x}{4} + y + \frac{z}{4} + \frac{1}{4} = 0 \Rightarrow x + 4y + z + 1 = 0 \tag{9}$$

$$-x - 1 = 0 \Rightarrow x + 1 = 0 \tag{10}$$

$$\frac{x}{4} + \frac{z}{4} + \frac{5}{4} = 0 \Rightarrow x + z + 5 = 0 \tag{11}$$

Subtracting equation (10) from equation (11) gives $z = -4$. With equation (11) we get $x = -1$ and then using equation (9) yields $y = 1$. The center of projection is the point $(-1, 1, -4)^T$.

c) The points on the projection plane are mapped to themselves by the perspective projection. Using the result from Exercise 9.1 a), the projection plane can be computed by:

$$\begin{pmatrix} 1 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{5}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \alpha \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

This gives the following equations:

$$\begin{aligned} x + \frac{y}{4} - \frac{z}{4} &= \alpha x \Rightarrow (1-\alpha)x + \frac{1}{4}y - \frac{1}{4}z = 0 \\ \frac{3}{4}y + \frac{1}{4}z &= \alpha y \Rightarrow (\frac{3}{4} - \alpha)y + \frac{1}{4}z = 0 \\ -\frac{y}{4} + \frac{5}{4}z &= \alpha z \Rightarrow -\frac{1}{4}y + (\frac{5}{4} - \alpha)z = 0 \\ \frac{y}{4} - \frac{z}{4} + 1 &= \alpha \Rightarrow \frac{1}{4}y - \frac{1}{4}z + 1 - \alpha = 0 \end{aligned}$$

By looking on the first and last equation one sees that α needs to be 1, which gives the projection plane $y - z = 0$.

9.4 Homogenous Lines*

Lets first compute the cross product between the two homogenous points, which yields:

$$l = p \times q = \begin{pmatrix} p_x \\ p_y \\ p_w \end{pmatrix} \times \begin{pmatrix} q_x \\ q_y \\ q_w \end{pmatrix} = \begin{pmatrix} p_y q_w - p_w q_y \\ p_w q_x - p_x q_w \\ p_x q_y - p_y q_x \end{pmatrix}$$

The two points lie on the line if the scalar products $p \cdot l = 0$ and $q \cdot l = 0$ are both zero.

$$\begin{aligned} p \cdot l &= \begin{pmatrix} p_x \\ p_y \\ p_w \end{pmatrix} \cdot \begin{pmatrix} p_y q_w - p_w q_y \\ p_w q_x - p_x q_w \\ p_x q_y - p_y q_x \end{pmatrix} \\ &= p_x(p_y q_w - p_w q_y) + p_y(p_w q_x - p_x q_w) + p_w(p_x q_y - p_y q_x) \\ &= p_x p_y q_w - p_x p_w q_y + p_y p_w q_x - p_y p_x q_w + p_w p_x q_y - p_w p_y q_x \\ &= 0 \\ q \cdot l &= \begin{pmatrix} q_x \\ q_y \\ q_w \end{pmatrix} \cdot \begin{pmatrix} p_y q_w - p_w q_y \\ p_w q_x - p_x q_w \\ p_x q_y - p_y q_x \end{pmatrix} \\ &= q_x(p_y q_w - p_w q_y) + q_y(p_w q_x - p_x q_w) + q_w(p_x q_y - p_y q_x) \\ &= q_x p_y q_w - q_x p_w q_y + q_y p_w q_x - q_y p_x q_w + q_w p_x q_y - q_w p_y q_x \\ &= 0 \end{aligned}$$