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COMPUTER GRAPHICS I
ASSIGNMENT 8

Submission deadline for the exercises: January 18, 2007

8.1 C^1 and G^1 continuity (25 Points)

a) Show that C^1 continuity implies G^1 continuity if the first derivative vectors have a length not equal to zero.

b) Consider the two spline segments $\gamma(t) = (t^2, t)$ and $\nu(t) = (t^3 + 4t + 1, 2t + 1)$, both defined on the interval $0 \leq t \leq 1$. Show that the curves join at $\gamma(1)$, $\nu(0)$ with C^0 and G^1, but not with C^1 continuity.

c) Show that the two spline segments $\gamma(t) = (t, t^2 - 2t)$ and $\nu(t) = (t + 1, t^2 - 1)$ are both C^1 and G^1 continuous where they join at $\gamma(1) = \nu(0)$.

8.2 Hermite Spline (25 Points)

The following cubic polynomial defines a spline curve in 3D:

$$p(t) = at^3 + bt^2 + ct + d \texttt{ with } a, b, c, d \in \mathcal{R}^3$$

a) Compute the coefficients $a, b, c,$ and d of the polynomials such that $p(0) = (0, 1, 0)^T$, $p(1) = (1, 0, 1)^T$, $\frac{dp}{dt}(0) = (1, 0, 1)^T$, $\frac{dp}{dt}(1) = (0, -1, 0)^T$. You have to write down the constraints and solve an system of equations.

b) Compute the same coefficients by using the Hermite basis.

8.3 DeCasteljau (25 Points)

a) You are given the Bezier spline $B(t)$ with the control points $b_0 = (1, 1)$, $b_1 = (2, 2)$, $b_3 = (5, 1)$, $b_4 = (5, 0)$. Use the DeCasteljau algorithm graphically to find the value $B\left(\frac{1}{2}\right)$ on the spline.

b) Now apply the algorithm numerically to get the exact point $B\left(\frac{1}{2}\right)$ on the spline.

8.4 Bezier Splines (25 Points)

Prove that two Bezier splines A and B are C^1 continous at the join point if $a_3 - a_2 = b_1 - b_0$.
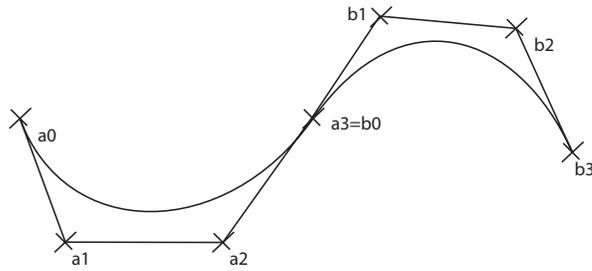
8.5 Invariance of Bezier Splines under Affine Transformations* (30 Points)

Prove that a Bezier spline is invariant under affine transformations, which means that transforming the control points is equivalent to transforming the complete curve.

Solutions

8.1 C^1 and G^1 continuity

a) Consider a curve f and g that join at point $f(0) = g(0)$ then the curves are C^1 at this point if $\frac{df}{dt}(0) = \frac{dg}{dt}(0)$ which implies $\frac{df}{dt}(0) = \alpha \frac{dg}{dt}(0)$ for $\alpha = 1$. As $\frac{df}{dt}(0) = \frac{dg}{dt}(0) \neq 0$ the derivatives show to the tangent direction $\Rightarrow G^1$ continuity.

b) The two curves join with C^0 continuity as $\gamma(1) = (1,1) = \nu(0)$. Now we derive $\gamma(t)$ and $\nu(t)$ by t: $\frac{d\gamma}{dt}(t) = (2t, 1)$ and $\frac{d\nu}{dt}(t) = (3t^2 + 4, 2)$. We compute the derivatives at the join point and get $\frac{d\gamma}{dt}(1) = (2,1)$ and $\frac{d\nu}{dt}(0) = (4,2)$. Thus $\frac{d\gamma}{dt}(1) \neq \frac{d\nu}{dt}(0)$ which means the curves are not C^1 continuous. But they are G^1 continuous as $2 \cdot \frac{d\gamma}{dt}(1) = \frac{d\nu}{dt}(0)$ and $\frac{d\nu}{dt}(0) \neq (0,0)$.

c) Again we compute the derivatives of both curves: $\frac{d\gamma}{dt}(t) = (1, 2t-2)$ and $\frac{d\nu}{dt}(t) = (1, 2t)$. Evaluating the derivative at the join position gives $\frac{d\gamma}{dt}(1) = (1,0)$ and $\frac{d\nu}{dt}(0) = (1,0)$. Thus the curves match with C^1 and G^1 continuity.

8.2 Hermite Spline

a) We build the first derivative of the spline segment first:

$$
\begin{aligned}
p(t) &= at^3 + 2t^2 + ct + d \\
p'(t) &= 3at^2 + 2bt + c
\end{aligned}
$$

The four constraints give the following equations:

$$
\begin{aligned}
p(0) &= a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = d = (0,1,0)^T & (1) \\
p'(0) &= 3a \cdot 0^3 + 2b \cdot 0^2 + c = c = (1,0,1)^T & (2) \\
p(1) &= a + b + (1,0,1)^T + (0,1,0)^T = a + b + (1,1,1)^T = (1,0,1)^T \Leftrightarrow a + b = (0,-1,0)^T & (3) \\
p'(1) &= 3a + 2b + (1,0,1)^T = (0,-1,0)^T \Leftrightarrow 3a + 2b = (-1,-1,-1)^T & (4)
\end{aligned}
$$

Then we combine the equations (3) and (4) like $(4) - 2 \cdot (3) \Leftrightarrow a = (1,-1,-1)^T$. We get b as $b = (1,-2,1)^T$.

The equation of the spline is:

$$
p(t) = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} t^3 + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
$$

b) An Hermite spline can be written as a linear combination $p(t) = b_0 \cdot p(0) + b_1 \cdot p(1) + b_2 \cdot p'(0) + b_3 \cdot p'(1)$ with the basis polynomials $b_0 = 2t^3 - 3t^2 + 1$, $b_1 = -2t^3 + 3t^2$, $b_2 = t^3 - 2t^2 + t$, and $b_3 = t^3 - t^2$. It follows:

$$
p(t) = (2t^3 - 3t^2 + 1) \cdot p(0) + (-2t^3 + 3t^2) \cdot p(1) + (t^3 - 2t^2 + t) \cdot p'(0) + (t^3 - t^2) \cdot p'(1)
$$

Rearranging the terms by the power of t yields:

$$
p(t) = (2p(0) - 2p(1) + p'(0) + p'(1))t^3 + (-3p(0) + 3p(1) - 2p'(0) - p'(1))t^2 + p'(0)t + p(0)
$$

Putting in the values for $p(0)$, $p(1)$, $p'(0)$, and $p'(1)$ to this equation gives the same result as we had before:

$$
p(t) = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} t^3 + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
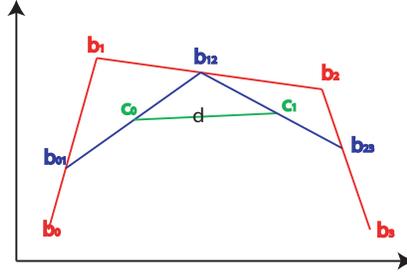$$

Figure 1: Graphical solution to 8.3

8.3 DeCasteljau

a) Graphical solution to 8.3 can be found in Figure 1.

b) Now we compute the point $d = B\left(\frac{1}{2}\right)$, which we just have constructed:

$$
\begin{aligned}
b_{01} &= \frac{1}{2}(1,1) + \frac{1}{2}(2,2) = \left(\frac{3}{2}, \frac{3}{2}\right) \\
b_{12} &= \frac{1}{2}(2,2) + \frac{1}{2}(5,1) = \left(\frac{7}{2}, \frac{3}{2}\right) \\
b_{23} &= \frac{1}{2}(5,1) + \frac{1}{2}(5,0) = \left(\frac{10}{2}, \frac{1}{2}\right) \\
c_0 &= \frac{1}{2}\left(\frac{3}{2}, \frac{3}{2}\right) + \frac{1}{2}\left(\frac{7}{2}, \frac{3}{2}\right) = \left(\frac{10}{4}, \frac{6}{4}\right) \\
c_1 &= \frac{1}{2}\left(\frac{7}{2}, \frac{3}{2}\right) + \frac{1}{2}\left(\frac{10}{2}, \frac{1}{2}\right) = \left(\frac{17}{4}, \frac{4}{4}\right) \\
d &= \frac{1}{2}\left(\frac{10}{4}, \frac{6}{4}\right) + \frac{1}{2}\left(\frac{17}{4}, \frac{4}{4}\right) = \left(\frac{27}{8}, \frac{10}{8}\right)
\end{aligned}
$$

Thus the point on the curve has the coordinates $B\left(\frac{1}{2}\right) = \left(\frac{27}{8}, \frac{10}{8}\right)$.

8.4 Bezier Splines

The two Bezier splines $A(t)$ and $B(t)$ can be represented using the Bezier basis functions (Bernstein polynomials):

$$A(t) = a_0(1-t)^3 + 3a_1(1-t)^2 t + 3a_2(1-t)t^2 + a_3 t^3$$

$$B(t) = b_0(1-t)^3 + 3b_1(1-t)^2 t + 3b_2(1-t)t^2 + b_3 t^3$$

The derivatives of these functions are:

$$A'(t) = -3a_0(1-t)^2 - 6a_1(1-t)t + 3a_1(1-t)^2 - 3a_2 t^2 + 6a_2(1-t)t + 3a_3 t^2$$

$$B'(t) = -3b_0(1-t)^2 - 6b_1(1-t)t + 3b_1(1-t)^2 - 3b_2 t^2 + 6b_2(1-t)t + 3b_3 t^2$$

The two curve segments join at $A(1) = B(0)$ with C^1 iff $A'(1) = B'(0)$:

$$A'(1) = -3a_2 + 3a_3$$

$$B'(0) = -3b_0 + 3b_1$$

Thus we must have $a_3 - a_2 = b_1 - b_0$ that the two splines join with C^1.

8.5 Invariance of Bezier Splines

Let $T(p) = Ap + B$ with $A \in \mathcal{R}^{3x3}$ and $B \in \mathcal{R}^3$ be an arbitrary affine transformation. An arbitrary Bezier spline $B(t)$ is given by the equation:

$$\texttt{Bezier}(b_0, b_1, b_2, b_3)(t) = B(t) = b_0(1-t)^3 + 3b_1(1-t)^2t + 3b_2(1-t)t^2 + b_3t^3$$

We transform $B(t)$ using the transformation $T(p)$ and get the following computation. Have in mind that $(1-t)^3 + 3(1-t)^2t + 3(1-t)t^2 + t^3 = 1$ for each t.

$$
\begin{aligned}
T(B(t)) &= T(\texttt{Bezier}(b_0, b_1, b_2, b_3)(t)) \\
&= A(b_0(1-t)^3 + 3b_1(1-t)^2t + 3b_2(1-t)t^2 + b_3t^3) + B \\
&= (Ab_0)(1-t)^3 + 3(Ab_1)(1-t)^2t + 3(Ab_2)(1-t)t^2 + (Ab_3)t^3 + \\
&\quad ((1-t)^3 + 3(1-t)^2t + 3(1-t)t^2 + t^3)B \\
&= (Ab_0 + B)(1-t)^3 + 3(Ab_1 + B)(1-t)^2t + 3(Ab_2 + B)(1-t)t^2 + (Ab_3 + B)t^3 \\
&= (T(b_0))(1-t)^3 + 3(T(b_1))(1-t)^2t + 3(T(b_2))(1-t)t^2 + (T(b_3))t^3 \\
&= \texttt{Bezier}(T(b_0), T(b_1), T(b_2), T(b_3))(t)
\end{aligned}
$$

Thus the affine transformation of the Bezier spline through the points b_0, b_1, b_2, and b_3 is the same as the Bezier spline through the transformed points $T(b_0), T(b_1), T(b_2)$, and $T(b_3)$.