UNIVERSITÄT DES SAARLANDES
Prof Dr. Philipp Slusallek
Lehrstuhl für ComputerGraphik
Sven Woop (woop@graphics.cs.uni-sb.de)
Stefan Popov (popov@graphics.cs.uni-sb.de)

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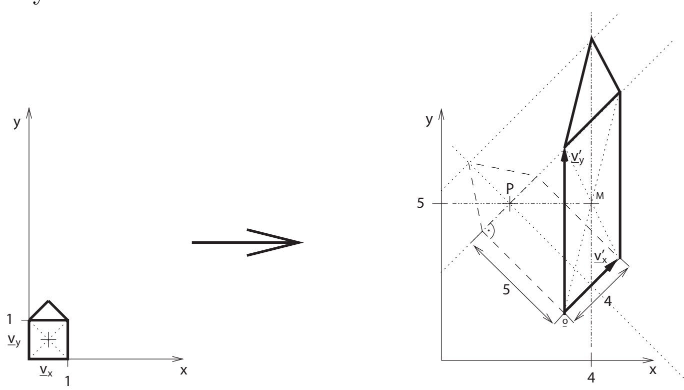
Computer Graphics I
Assignment 7

Submission deadline for the exercises: Thursday, 11th January 2007

7.1 Transformations (50 Points)

In the picture below the left house should be transformed into the house on the right. The point M is at $(4, 5)$ and lines that look to be parallel are parallel. Please specify the complete transformation matrix as a sequence of primitive transformations (there's no need to calculate the final matrix). Do not guess any numbers.



7.2 Affine Spaces (20 Points)

Prove that the set of points $A = \{(x, y, z, w) \in R^4 \mid w = 1\}$ is an affine space. What is the associated vector space? You do *not* have to show that the associated vector space is a vector space. What is the difference between a point and a vector in that affine space?

Definition of an affine space: An affine space consists of a set of points P, an associated vector space V and an operation $+ \in P \times V \to P$ that fulfills the following axioms:

(1) `for` $p \in P$ `and` $v, w \in V : (p + v) + w = p + (v + w)$

(2) `for` $p, q \in P$ `there exists a unique` $v \in V$ `such that:` $p + v = q$

7.3 Rotations (30 Points)

Show that an arbitrary rotation around the origin in 2D can be represented by a combination of a shearing in y, a scaling in x and y and a shearing in x in this order. You have to derive the shearing and scaling matrices to an arbitrary rotation T.

$$T = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

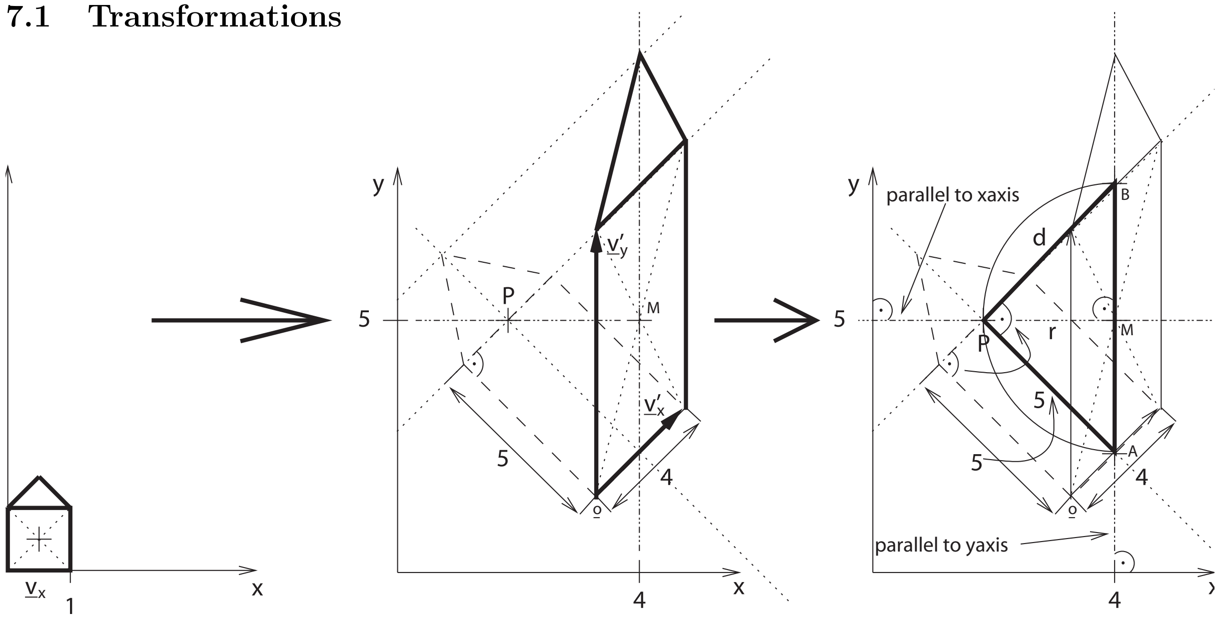
7.4 Affine Spaces* (20 Points)

a) Prove that for any affine space $p + 0 = p$ is true. Use only the axioms of the definition 7.2 of an affine space.

b) A property of affine spaces is that a subtraction $- \in P \times P \to V$ exists that gives the connecting vector between two points. The definition of an affine space from Exercise 7.2 is missing such an axiom. How can you derive this operation from the axioms of an affine space?

Solutions

7.1 Transformations



Several transformations have to be performed: first the house is scaled by matrix S, then it is sheared by matrix Sh, after that it is rotated by matrix R and finally it is translated with matrix T. Scaling is simple, since it can be seen directly in the figure above:

$$S = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By looking at the figure above we see that we have to shear in x-direction only and the amount of shearing can be derived by looking at the only angle which was given in the figure. Let's take a look at triangle (P, A, B): The angle between $\overline{PA}$ and $\overline{PB}$ is 90 degrees as given in the exercise. This means that the point P is on Thale's circle *(dt. der Thales Kreis)* over $\overline{AB}$, with $r = \overline{PM}$ its radius, since M divides $\overline{AB}$ in the middle due to construction of M. Therefore we know that the length of $\overline{MB}$ is equal to the length of r. Let's take a look at the triangle (P, M, B): Since two sides are equal in length, the triangle is isosceles *(dt.[math.]: gleichschenklig)*, which together with the angle between $\overline{MP}$ and $\overline{MB}$ gives us the angle between $\overline{BP}$ and $\overline{BM}$ which is 45 degrees. Since the same holds for triangle (P, A, M) we know that also triangle (P, A, B) is isosceles and therefore $d = \overline{PB}$ is equal in length to $\overline{PA}$ which was given in the exercise to have length 5. Thus we know that we have to shear for 5 unit lengths a point which is at y=5. This means that for each unit in direction of the y-axis we shear by one unit length in the direction of the positive x-axis. Using the notation of the lecture we write $H = (1, 0, 0, 0, 0, 0)$ and we get:

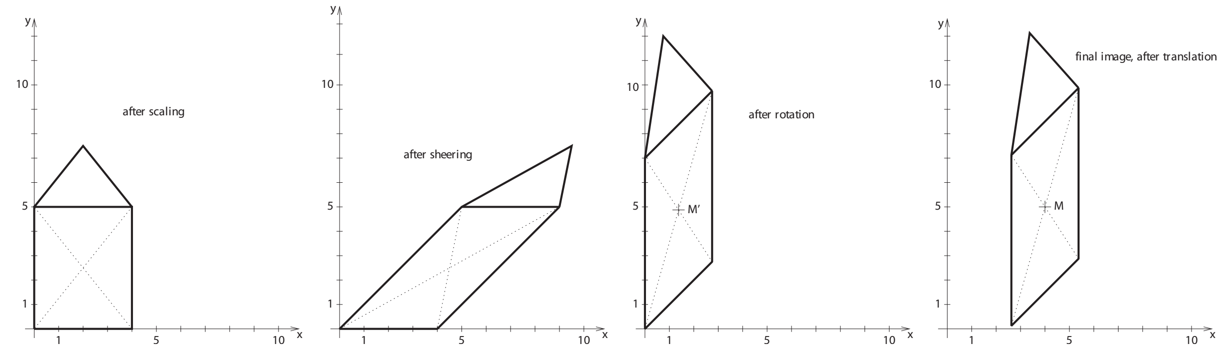
$$Sh = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now to the rotation: We have to rotate the house so that the left side wall matches the y-axis. Before we sheared our house the left side wall matched the y-axis. So what we have to do is simply to rotate the house by $arctan(\frac{5}{5})$ which is simply 45 degrees. Therefore:

$$R = \begin{pmatrix} cos(\frac{\pi}{4}) & -sin(\frac{\pi}{4}) & 0 \\ sin(\frac{\pi}{4}) & cos(\frac{\pi}{4}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Finally we have to translate the house so that point M' matches point M (see figure below). The position of M was given in the exercise and everything we need to calculate the position of M' after the scaling, shearing and rotation. Applying these transformations to M gives $M' = (1.4, 4.9)$ and therefore

$$T = \begin{pmatrix} 1 & 0 & 2.6 \\ 0 & 1 & 0.1 \\ 0 & 0 & 1 \end{pmatrix}$$



7.2 Affine Spaces

The set $V = \{(x, y, z, w) \in R^4 \mid w = 0\}$ is the associated vector space. This is be shown by proving (1) and (2) in the following:

a) Associativity: Let be $p \in P, v, w \in V$:

$$\begin{aligned}
(p + v) + w &= ((p_x, p_y, p_z, 1) + (v_x, v_y, v_z, 0)) + (w_x, w_y, w_z, 0) \\
&= (p_x + v_x, p_y + v_y, p_z + v_z, 1 + 0) + (w_x, w_y, w_z, 0) \\
&= ((p_x + v_x) + w_x, (p_y + v_y) + w_y, (p_z + v_z) + w_z, (1 + 0) + 0) \\
&= (p_x + (v_x + w_x), p_y + (v_y + w_y), p_z + (v_z + w_z), 1 + (0 + 0)) \\
&= (p_x + v_x, p_y + v_y, p_z + v_z, 1) + (v_x + w_x, v_y + w_y, v_z + w_z, 0 + 0) \\
&= (p_x + v_x, p_y + v_y, p_z + v_z, 1) + ((v_x, v_y, v_z, 0) + (w_x, w_y, w_z, 0)) \\
&= p + (v + w)
\end{aligned}$$

b) Unique connection vector: Let be $p, q \in P, v \in V$:

$$\begin{aligned}
& p + v = q \\
\Leftrightarrow\ & (p_x, p_y, p_z, 1) + (v_x, v_y, v_z, 0) = (q_x, q_y, q_z, 1) \\
\Leftrightarrow\ & (p_x + v_x, p_y + v_y, p_z + v_z, 1) = (q_x, q_y, q_z, 1) \\
\Leftrightarrow\ & p_x + v_x = q_x \wedge p_y + v_y = q_y \wedge p_z + v_z = q_z \wedge 1 = 1 \\
\Leftrightarrow\ & v_x = q_x - p_x \wedge v_y = q_y - p_y \wedge v_z = q_z - p_z \wedge 0 = 0 \\
\Leftrightarrow\ & (v_x, v_y, v_z, 0) = (q_x - p_x, q_y - p_y, q_z - p_z, 1 - 1) \in V \\
\Leftrightarrow\ & v = q - p
\end{aligned}$$

There exists exactly one such vector $v = (q_x - p_x, q_y - p_y, q_z - p_z, 0)$ as the subtraction in $\mathcal{R}$ is well defined.

7.3 Rotations

An arbitrary rotation by the angle ϕ around the origin is given by:

$$T = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

We try to represent this rotation by the combination of a shearing in y, followed by a scaling, and a shearing in x again:

$$T = \begin{pmatrix} 1 & s_x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_x & 0 \\ 0 & \lambda_y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_y & 1 \end{pmatrix} = \begin{pmatrix} \lambda_x & s_x\lambda_y \\ 0 & \lambda_y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_y & 1 \end{pmatrix} = \begin{pmatrix} \lambda_x + s_x s_y \lambda_y & s_x\lambda_y \\ \lambda_y s_y & \lambda_y \end{pmatrix}$$

This yields the following system of equations:

$$\begin{aligned} \lambda_x + s_x s_y \lambda_y &= \cos\phi && (1) \\ s_x\lambda_y &= -\sin\phi && (2) \\ \lambda_y s_y &= \sin\phi && (3) \\ \lambda_y &= \cos\phi && (4) \end{aligned}$$

We get directly $\lambda_y = \cos\phi$, which together with equation (2) and (3) gives us $s_x = -\tan\phi$ and $s_y = \tan\phi$. Inserting to equation (1) yields $\lambda_x - \tan^2\phi\cos\phi = \cos\phi \Leftrightarrow \lambda_x = \cos\phi + \frac{\sin^2\phi}{\cos\phi} = \frac{1}{\cos\phi}(\sin^2 + \cos^2) = \frac{1}{\cos\phi}$. Thus the complete composition for the rotation is:

$$T = \begin{pmatrix} 1 & -\tan\phi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\cos\phi} & 0 \\ 0 & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tan\phi & 1 \end{pmatrix}$$

7.4 Affine Spaces*

a) Proof:

$$\begin{aligned} & q = q + a \\ \overset{0\in V}{\Leftrightarrow} \quad & q + 0 = (q + a) + 0 \\ \overset{(1)}{\Leftrightarrow} \quad & q + 0 = q + (a + 0) \\ \overset{\texttt{vector space}}{\Leftrightarrow} \quad & q + 0 = q + a \\ \overset{\texttt{(2) unique}}{\Leftrightarrow} \quad & 0 = a \end{aligned}$$

b) The operation $- \in P \times P \to V : p - q = v$ `such that` $p + v = q$ is well defined as v is unique by axiom (2). By definition $p - q$ gives the connection vector.