



27TH NOVEMBER 2006

## COMPUTER GRAPHICS I ASSIGNMENT 5

**Submission deadline for the exercises:** Thursday, 7th December 2006

### 5.1 Fourier Transformation (30 Points)

Show that the Fourier transformation of the box function  $B(x)$  is a *sinc* type function. The sinc function is defined as  $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$  and a definition of the Fourier transform can be found in Exercise 5.5.

$$B(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ 1 & \text{for } -1 < x < 1 \\ 0 & \text{for } 1 \leq x \end{cases}$$

### 5.2 Sampling Theory (10 + 10 Points)

Let  $f(x)$  be an infinite signal that fulfills the Nyquist property, thus the highest frequency of the signal is smaller than  $\frac{1}{2T}$  if  $T$  is the sampling distance. Consider a regular sampling  $f_S(x)$  of  $f(x)$  with sample distance  $T$ .

- Is an exact signal reconstruction of  $f(x)$  possible? If so, why?
- How has the reconstruction to be performed in image and Fourier space?

### 5.3 Antialiasing (10 + 10 Points)

- Explain what sampling of a continuous signal means in signal and Fourier space. Further explain what aliasing of a sampled signal means in signal and Fourier space.
- Consider an infinite signal  $f(x)$  and a regular sampling  $f_S(x)$  with sampling distance  $d$  that shows no aliasing artefacts. The sampling distance  $d$  is now increased step by step until the first aliasing artefacts occur.

How can we best get an *aliasing-free* sampling from these samples  $f_S$  again? The sampling distance should stay  $d$  and as much useful information as possible should be recovered.

Describe the filter procedure in Fourier and signal space. You do not have to derive the exact filter kernels (but you can of course).

### 5.4 Triangle Filter (30 Points)

Show that reconstructing a signal that is sampled at sampling distance 1 with the triangle filter  $T(x)$  is equivalent of performing linear interpolation.

$$T(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ x + 1 & \text{for } -1 < x < 0 \\ -x + 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{for } 1 \leq x \end{cases}$$

## 5.5 Duality of Multiplication and Convolution\* (20 Points)

The convolution of a function  $f(t)$  with a second function  $g(t)$  is defined as:

$$(f \otimes g)(t) = \int_{-\infty}^{+\infty} f(\tau) \cdot g(t - \tau) d\tau$$

The multiplication of two function is defined as the point-wise multiplication:

$$(f \cdot g)(t) = f(t) \cdot g(t)$$

The transformation of a signal  $f(x)$  to Fourier space is given by:

$$F(k) = \int_{-\infty}^{+\infty} f(x) \cdot e^{-2\pi i k x} dx$$

We call  $\mathcal{F}$  the operator mapping  $f$  to Fourier space:  $\mathcal{F}f = F$ . Show that convolving in signal space is the same as multiplication in Fourier space:

$$\mathcal{F}[f \otimes g] = \mathcal{F}[f] \cdot \mathcal{F}[g]$$

# Solutions

## 5.1 Fourier Transformation

The bounds of the box lead to the integral ranges  $[-1, 1]$  for the Fourier transformation.

$$\begin{aligned} F(k) &= \int_{-1}^{+1} \cos(-2\pi xk) + i\sin(-2\pi xk) dx \\ &= \left. \frac{\sin(-2\pi kx)}{-2\pi k} + \frac{i \cos(-2\pi kx)}{2\pi k} \right|_{-1}^{+1} \\ &= \frac{\sin(-2\pi k)}{-2\pi k} + \frac{i \cos(-2\pi k)}{2\pi k} + \frac{\sin(2\pi k)}{2\pi k} - \frac{i \cos(2\pi k)}{2\pi k} \\ &= \frac{2 \sin(2\pi k)}{2\pi k} \\ &= 2 \operatorname{sinc}(2k) \end{aligned}$$

Which is a sinc function as expected.

## 5.2 Sampling Theory

- As the spectrum is in some sense symmetric it ranges from  $-\frac{1}{2T}$  to  $\frac{1}{2T}$  and has the width of  $\frac{1}{T}$ . An exact signal reconstruction of  $f(x)$  is thus possible as the Fourier spectrum does not overlap itself after sampling, which means replication of the spectrum each  $\frac{1}{T}$ . All information is still present in the spectrum.
- To get the original spectrum again, we have to multiply the Fourier spectrum with the box filter  $B_{\frac{1}{2T}}(k)$ , which means filtering with the Fourier synthesis of this box function, which is the sinc-kernel  $2\frac{1}{2T}\operatorname{sinc}(2x\frac{1}{2T}) = \frac{1}{T}\operatorname{sinc}(\frac{x}{T})$  in signal space. The Fourier synthesis of a box function gets the same function as the Fourier transform of it, which we computed in the last exercise.

## 5.3 Antialiasing

- Sampling a signal in signal space at sample positions with distance  $d$  means multiplying the signal with the comb function with comb distance  $d$  in signal space. In Fourier space this is equivalent to convolve the spectrum with a comb function with comb distance  $1/d$ , or in other words the spectrum is replicated each  $1/d$ . If the spectra overlap by this replication aliasing occurs.
- The regions at which the spectrum overlapped can not be reconstructed again, which means that some information is lost. The only possibility to get an aliasing free signal is to cut the overlapping parts out of the spectrum of the sampled signal. If the overlapping in Fourier space is  $\Delta f$  at both sides of the spectrum then in Fourier space one has to multiply with the convolution of a comb function with comb distance  $\frac{1}{d}$  and the box  $B_{\frac{1}{2d}-\Delta f}(x)$  which is the same as filtering the signal with the sampling at distance  $d$  of the sinc function:

$$2 \left( \frac{1}{2d} - \Delta f \right) \operatorname{sinc} \left( 2 \left( \frac{1}{2d} - \Delta f \right) x \right) = \left( \frac{1}{d} - 2\Delta f \right) \operatorname{sinc} \left( \left( \frac{1}{d} - 2\Delta f \right) x \right)$$

## 5.4 Reconstruction

Let  $p_a$  be the pixel on the left to the center of convolution  $x$  and  $\lambda = x - p_a$  it's distance from  $x$ . Further  $p_b$  is the value of the pixel to the right of the center of convolution with distance  $1 - \lambda$ . The definition of convolution gives us the equation:

$$(f \otimes T)(x) = \int_{-\infty}^{+\infty} f(\tau) \cdot T(x - \tau) d\tau$$

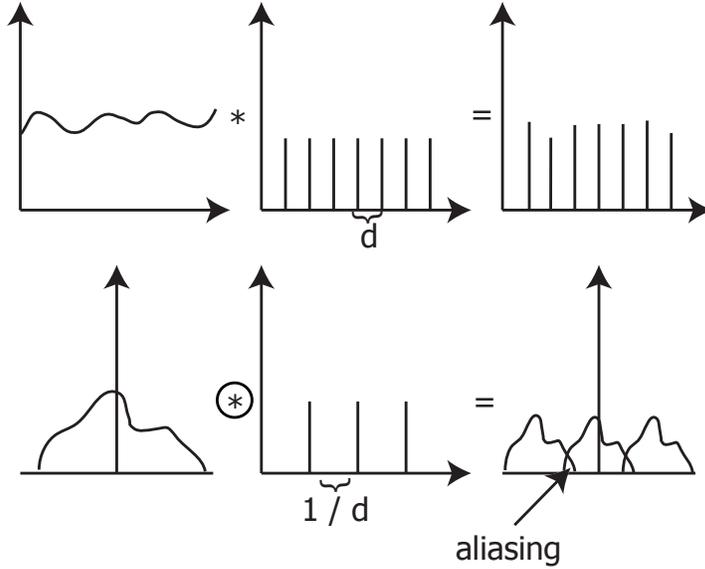


Figure 1: In Fourier space aliasing means an overlapp of the spectra after the replication due to the sampling.

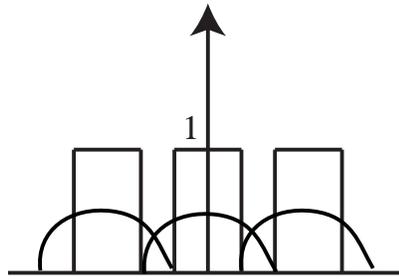


Figure 2: Reconstruction of alias free images via cutting of overlapping parts in Fourier space

As  $T$  is zero for values smaller  $-1$  and greater  $1$  we can ignore the parts of the integral with  $x - \tau < -1 \Leftrightarrow \tau > x + 1$  and  $x - \tau > 1 \Leftrightarrow \tau < x - 1$ . Thus samples  $p$  to the left of  $p_a$  have no impact to the result as their distance to  $x$  is  $x - p > 1 + \lambda$ , thus  $p < x - 1 - \lambda < x - 1$ . Similar we get that samples to the right of  $p_b$  have no influence on the result.

Because the samples are multiples of a shifted delta function, the integral

$$(f \otimes T)(x) = \int_{x-1}^{x+1} f(\tau) \cdot T(x - \tau) d\tau$$

becomes the sum  $f(x - \lambda) \cdot T(x - (x - \lambda)) + f(x + (1 - \lambda)) \cdot T(x - (x + (1 - \lambda)))$  as  $f$  is unequal zero only at  $p_a$  and  $p_b$  in the interval  $[x - 1, x + 1]$ . As  $f(x - \lambda) = f(p_a)$ ,  $f(x + (1 - \lambda)) = f(x + 1 - x + p_a) = f(p_a + 1) = f(p_b)$  we get  $f(p_a) \cdot T(\lambda) + f(p_b) \cdot T(\lambda - 1)$ . As  $\lambda$  is greater zero,  $T(\lambda) = -\lambda + 1$  and similar  $\lambda - 1$  is smaller than zero, thus  $T(\lambda - 1) = \lambda$ . This gives us the result  $f(p_a) \cdot (1 - \lambda) + f(p_b) \cdot \lambda$ , which is a linear interpolation.

## 5.5 Duality of Multiplication and Convolution\*

$$\begin{aligned} \mathcal{F}[f \otimes g] &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\tau) \cdot g(t - \tau) \cdot e^{-2\pi i k x} d\tau dx \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(\tau) \cdot e^{-2\pi i k \tau}) \cdot (g(x - \tau) \cdot e^{-2\pi i k (x - \tau)}) d\tau dx \end{aligned}$$

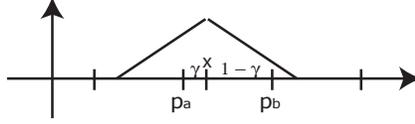


Figure 3: Triangle kernel used for reconstruction.

$$\begin{aligned}
 &= \int_{-\infty}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} (f(\tau) \cdot e^{-2\pi i k \tau} d\tau)}_{\text{constant in } x} \cdot (g(x - \tau) \cdot e^{-2\pi i k(x-\tau)}) dx \\
 &= \left( \int_{-\infty}^{+\infty} f(\tau) \cdot e^{-2\pi i k \tau} d\tau \right) \cdot \left( \int_{-\infty}^{+\infty} g(x - \tau) \cdot e^{-2\pi i k(x-\tau)} dx \right) \\
 &= \mathcal{F}[f] \cdot \mathcal{F}[g]
 \end{aligned}$$