Computer Graphics I

Assignment 9

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9.1 Homogenous Coordinates

a. Let $\tilde{m} = (x, y, z, w)$ be an homogenous point. Then this point represents the point $m = \left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)$ in euclidean coordinates, except for $w = 0$ (these points are located at infinity).
Now we multiply our point $\tilde{m}$ with an arbitrary $\alpha \neq 0$:

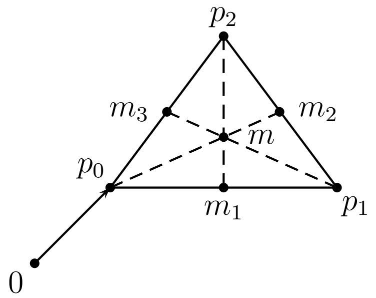
$$\begin{aligned} \tilde{m}' = \alpha\tilde{m} &= \alpha \cdot (x, y, z, w) \\ &= (\alpha x, \alpha y, \alpha z, \alpha w) \end{aligned}$$

To get the euclidean point, that is represented by this homogenous point $\tilde{m}'$ we divide the first three components again by the forth:

$$m' = \left(\frac{\alpha x}{\alpha w}, \frac{\alpha y}{\alpha w}, \frac{\alpha z}{\alpha w}\right) = \left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right) = m$$

$\square$

b. Figure:



To get the center $m = (m_x, m_y, m_z)$ between three points $p_0 = (a_0, b_0, c_0)$, $p_1 = (a_1, b_1, c_1)$ and $p_2 = (a_2, b_2, c_2)$ in euclidean coordinates we make the following computations:

$$m_1 = \frac{p_0 + p_1}{2}$$

By the properties of the barycenter in a triangle we get:

$$\begin{aligned} m &= m_1 + \frac{1}{3} \cdot (p_2 - m_1) \\ &= \frac{p_0 + p_1}{2} + \frac{2p_2 - p_0 - p_1}{3 \cdot 2} \\ &= \frac{3p_0 + 3p_1 + 2p_2 - p_0 - p_1}{6} \\ &= \frac{2p_0 + 2p_1 + 2p_2}{6} \\ &= \frac{p_0 + p_1 + p_2}{3} \end{aligned}$$

Now we'll make the component wise addition of the three points in homogenous coordinates:

$$\tilde{m} = \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ 1 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ 1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_0 + a_1 + a_2 \\ b_0 + b_1 + b_2 \\ c_0 + c_1 + c_2 \\ 3 \end{pmatrix}$$

In euclidean coordinates $\tilde{m}$ is:

$$\frac{1}{3} \cdot \begin{pmatrix} a_0 + a_1 + a_2 \\ b_0 + b_1 + b_2 \\ c_0 + c_1 + c_2 \end{pmatrix} = \frac{p_0 + p_1 + p_2}{3} = m$$

□

c. Define the addition $\oplus$ between two homogenous points $\tilde{a}, \tilde{b}$ as follows:

$$\tilde{a} \oplus \tilde{b} = \begin{pmatrix} a_x b_w + b_x a_w \\ a_y b_w + b_y a_w \\ a_z b_w + b_z a_w \\ b_w a_w \end{pmatrix}$$

The corresponding euclidean point for $\tilde{a} \oplus \tilde{b}$ is then

$$\frac{1}{b_w a_w} \cdot \begin{pmatrix} a_x b_w + b_x a_w \\ a_y b_w + b_y a_w \\ a_z b_w + b_z a_w \end{pmatrix}$$

The corresponding euclidean point to $\tilde{a}$ is $a = \left(\frac{a_x}{a_w}, \frac{a_y}{a_w}, \frac{a_z}{a_w} \right)^T$ (with $a_w \neq 0$). For $\tilde{b}$ we obtain b analogously.
Then we get for the standard addition:

$$a + b = \begin{pmatrix} \frac{a_x}{a_w} + \frac{b_x}{b_w} \\ \frac{a_y}{a_w} + \frac{b_y}{b_w} \\ \frac{a_z}{a_w} + \frac{b_z}{b_w} \end{pmatrix} = \begin{pmatrix} \frac{a_x b_w + b_x a_w}{a_w b_w} \\ \frac{a_y b_w + b_y a_w}{a_w b_w} \\ \frac{a_z b_w + b_z a_w}{a_w b_w} \end{pmatrix}$$

Now it is obvious that $(\tilde{a} \oplus \tilde{b})$ is the corresponding homogenous point to the euclidean one $(a + b)$.

9.4 Homogenous Lines in 2D*

Claim:

$$\begin{pmatrix} p_x \\ p_y \\ p_w \end{pmatrix} \times \begin{pmatrix} q_x \\ q_y \\ q_w \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where $ax + by + c = 0$ is the line through p and q.

Proof:

$$\begin{aligned} a &= p_y q_w - p_w q_y \\ b &= p_w q_x - p_x q_w \\ c &= p_x q_y - p_y q_x \end{aligned}$$

Test if p is on the line:

$$\begin{aligned} & (p_y q_w - p_w q_y)\frac{p_x}{p_w} + (p_w q_x - p_x q_w)\frac{p_y}{p_w} + (p_x q_y - p_y q_x) \\ = \; & \frac{p_x p_y q_w}{p_w} - p_x q_y + p_y q_x - \frac{p_x p_y q_w}{p_w} + p_x q_y - p_y q_x \\ = \; & 0 \end{aligned}$$

Test if q is on the line:

$$\begin{aligned} & (p_y q_w - p_w q_y)\frac{q_x}{q_w} + (p_w q_x - p_x q_w)\frac{q_y}{q_w} + (p_x q_y - p_y q_x) \\ = \; & p_y p_x - \frac{p_w q_y q_x}{q_w} + \frac{p_w q_x q_y}{q_w} - p_x q_y + p_x q_y - p_y q_x \\ = \; & 0 \end{aligned}$$

$\square$