Computer Graphics I

Assignment 6

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6.1 Gamma correction

We look at the following cases:

- $x = 0.00000000b$
- $x = 0.00000001b$
- $x = 0.11111110b$
- $x = 0.11111111b$

If γ equals 1, we have no problem, because the result is exactly the same color value. But – assuming the worst-case – in the case that γ equal 2.5, 8 bits actually don't suffice. This is show by the following calculation:

$$\begin{aligned}
f(0.00000000b) &= 0.0b \\
f(0.00000001b) &= 0.000000000000000000001111111111b \\
f(0.11111110b) &= 0.1111101100b \\
f(0.11111111b) &= 0.1111110110b
\end{aligned}$$

If we count the bits, we see that there are 30 bits needed to calculate the second result which means that we need 30 bits to accurately represent the correct signal after the gamma correction.

6.2 Transforms in colorspace

a. To visualize the problem, we imagine the color triangle to be a tetrahedron, with the base on the triangle itself (now residing on level 0), but the white point raised to 1 (as depicted by figure 1, left). The points of the colorspace now reside on the outer boundaries of this tetrahedron. It can now easily be seen that a shift of the white point results in a modification of the top corner of the tetrahedron in a plane parallel to its base, while the base points are left where they are (cf. Figure 1, right).

The vector $\underline{s}$ drawn in the figure represents the difference in each color value, which can be computed using the barycentric coordinates of the color space. By intercept theorems one sees that any point is shifted along this vector, but scaled by the "height" of the respective point.

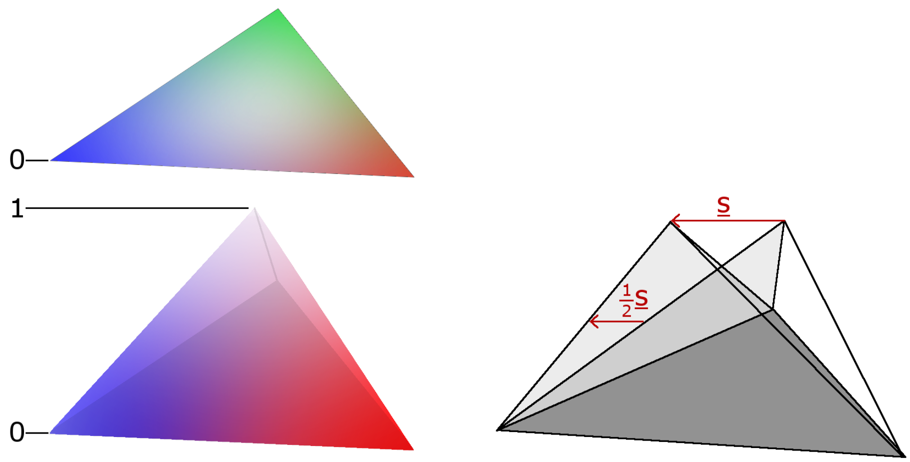

Figure 1: **Left:** From the color triangle to a tetrahedron - the white point is lifted by 1 along an axis which is parallel to the triangle's normal vector. **Right:** The shift vector is scaled by the "height" of the point being shifted.

Mathematically, the height can be expressed as

$$h(r, g, b) = \frac{r + g + b - 1}{2}$$

hence the incoming RGB signal $(i_r, i_g, i_b)^T$ must be translated to $(o_r, o_g, o_b)^T$ by

$$\begin{aligned} \begin{pmatrix} o_r \\ o_g \\ o_b \end{pmatrix} &= h(i_r, i_g, i_b) \cdot \begin{pmatrix} \underline{s}_r & 0 & 0 \\ 0 & \underline{s}_g & 0 \\ 0 & 0 & \underline{s}_b \end{pmatrix} \cdot \begin{pmatrix} i_r \\ i_g \\ i_b \end{pmatrix} \\ &= \frac{i_r, i_g, i_b - 1}{2} \cdot \begin{pmatrix} \underline{s}_r & 0 & 0 \\ 0 & \underline{s}_g & 0 \\ 0 & 0 & \underline{s}_b \end{pmatrix} \cdot \begin{pmatrix} i_r \\ i_g \\ i_b \end{pmatrix} \end{aligned}$$

b. Let M_1 be the transformation matrix from sRGB to CIE-XYZ (the computation is depicted in Exercise 6.3) and M_2 the transformation matrix for the colorspace with white point $(w_x, w_y) = (0.400, 0.330)$. M_2 can be computed as follows: Compute the C_X:

$$\begin{pmatrix} 0.6400 & 0.3000 & 0.1500 \\ 0.3300 & 0.6000 & 0.0600 \\ 0.0300 & 0.1000 & 0.7900 \end{pmatrix} \begin{pmatrix} C_r \\ C_g \\ C_b \end{pmatrix} = \begin{pmatrix} \frac{0.4000}{0.3300} \\ 1 \\ \frac{0.2700}{0.3300} \end{pmatrix}$$

Using Maple, the equation system can be solved with

$$\begin{aligned} C_r &= 1.277838627 \\ C_g &= 0.8762322017 \\ C_b &= 0.8762322015 \end{aligned}$$

Hence, M_2 can be calculated as:

$$M_2 \approx \begin{pmatrix} 0.8178167213 & 0.2628696605 & 0.1314348302 \\ 0.4216867469 & 0.5257393210 & 0.0525739321 \\ 0.0383351588 & 0.0876232202 & 0.6922234392 \end{pmatrix}$$

To transform a value (a_1, b_1) of the first colorspae into (a_2, b_2) in the second one, the following steps must be applied:

(a) Compute X_1, Y_1, Z_1 as

$$X_1 = \frac{a_1}{b_1}, \quad Y_1 = 1, \quad Z_1 = \frac{1 - a_1 - b_1}{b}.$$

(b) Transform X_1, Y_1, Z_1 into RGB values using M_1^{-1} and retransform these values into X_2, Y_2, Z_2 using M_2:

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} = M \cdot \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix},$$

where

$$\begin{aligned} M :&= M_2 \cdot M_1^{-1} \\ &\approx \begin{pmatrix} 2.403046396010846 & -0.790972406319101 & -0.257925781835294 \\ 0.860029272312178 & 0.327251898802813 & -0.132841273188141 \\ 0.077823293962256 & -0.035755146854610 & 0.716187322029735 \end{pmatrix}. \end{aligned}$$

(c) Compute a_2, b_2 as

$$a_2 = \frac{X_2}{X_2 + Y_2 + Z_2}$$
$$b_2 = \frac{Y_2}{X_2 + Y_2 + Z_2}$$

6.3 Color Space

We have the following:

$$\begin{aligned}
x = \frac{X}{X+Y+Z} &\Leftrightarrow xX + xY + xZ = X \\
&\Leftrightarrow X \cdot (x-1) = -xY - xZ \\
&\Leftrightarrow X = \frac{xY + xZ}{1-x} \quad (1)
\end{aligned}$$

$$\begin{aligned}
y = \frac{Y}{X+Y+Z} &\Leftrightarrow yX + yY + yZ = Y \\
&\Leftrightarrow Z = \frac{Y - yX - yY}{y} \quad (2)
\end{aligned}$$

Now, we put equation (2) into equation (1):

$$\begin{aligned}
X &= \frac{xY + x \cdot \left(\frac{Y - yX - yY}{y}\right)}{1-x} \\
&= \frac{xyY + xY - xyX - xyY}{y - xy} \\
\Leftrightarrow X + \frac{xyX}{y-xy} &= \frac{xY}{y-xy} \\
\Leftrightarrow X \cdot \left(\frac{y - xy + xy}{y - xy}\right) &= \frac{xY}{y-xy} \\
\Leftrightarrow X &= \frac{xY \cdot (y-xy)}{(y-xy) \cdot y} = \frac{xY}{y}
\end{aligned}$$

We put the last equation in equation (2):

$$Z = \frac{Y - y \cdot \frac{xY}{y} - yY}{y} = \frac{Y \cdot (1 - x - y)}{y}$$

For (X_w, Y_w, Z_w) we have

$$\begin{aligned} Y_w &= 1 \\ X_w &= \frac{x_w}{y_w} \\ Z_w &= \frac{1 - x_w - y_w}{y_w} \end{aligned}$$

Compare the C_X:

$$\begin{pmatrix} x_r & x_g & x_b \\ y_r & y_g & y_b \\ (1 - x_r - y_r) & (1 - x_g - y_g) & (1 - x_b - y_b) \end{pmatrix} \begin{pmatrix} C_r \\ C_g \\ C_b \end{pmatrix} = \begin{pmatrix} \frac{x_w}{y_w} \\ 1 \\ \frac{1 - x_w - y_w}{y_w} \end{pmatrix}$$

For sRGB we have:

$$\begin{pmatrix} 0.6400 & 0.3000 & 0.1500 \\ 0.3300 & 0.6000 & 0.0600 \\ 0.0300 & 0.1000 & 0.7900 \end{pmatrix} \begin{pmatrix} C_r \\ C_g \\ C_b \end{pmatrix} = \begin{pmatrix} \frac{0.3127}{0.3290} \\ 1 \\ \frac{0.3583}{0.3290} \end{pmatrix}$$

Using Maple, we can calculate the values for C_r, C_g, C_b:

$$\begin{aligned} C_r &= 1.191947798 \\ C_g &= 0.6443606239 \\ C_b &= 1.203205256 \end{aligned}$$

Now, we get as transformation matrix the following:

$$\begin{aligned} M &:= \begin{pmatrix} x_r C_r & x_g C_g & x_b C_b \\ y_r C_r & y_g C_g & y_b C_b \\ (1 - x_r y_r) C_r & (1 - x_g - y_g) C_g & (1 - x_b - y_b) C_b \end{pmatrix} \\ &\approx \begin{pmatrix} 0.4123907993 & 0.3575843394 & 0.1804807884 \\ 0.2126390059 & 0.7151686788 & 0.07219231536 \\ 0.01933081872 & 0.1191947798 & 0.9505321522 \end{pmatrix} \end{aligned}$$

Finally, we transform the sRGB color $(1, 1, 0)$ in the CIE-XYZ color space:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = M \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 0.7699751387 \\ 0.9278076847 \\ 0.1385255985 \end{pmatrix}$$

Computing sRGB color $(1, 1, 0)$ in the CIE-xy color space:

$$x = \frac{X}{X+Y+Z} = 0.4193059997$$
$$y = \frac{Y}{X+Y+Z} = 0.5052570002$$
$$z = 1 - x - y = 0.0754370001$$