Computer Graphics
Assignment 1

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Exercise 1.1

$$\begin{aligned}
focus &= |\underline{dir}| = \sqrt{\underline{dir}.x^2 + \underline{dir}.y^2 + \underline{dir}.z^2} \\
aspect\ ratio &= \frac{resX}{resY} \\
\underline{y} &= focus \cdot \tan\left(\frac{angle}{2}\right) \cdot \frac{\underline{dir} \times \underline{x}}{|\underline{dir} \times \underline{x}|} \\
\underline{x} &= (aspect\ ratio \cdot focus) \cdot \tan\left(\frac{angle}{2}\right) \cdot \frac{\underline{dir} \times \underline{up}}{|\underline{dir} \times \underline{up}|} \\
\underline{f} &= focus \cdot \frac{\underline{dir}}{|\underline{dir}|}
\end{aligned}$$

Now one has to use the formula from the lecture:

$$\underline{ray.dir} = \underline{f} + 2\left(\frac{X + 0.5}{resX} - \frac{1}{2}\right) \cdot \underline{x} + 2\left(\frac{Y + 0.5}{resY} - \frac{1}{2}\right) \cdot \underline{y}$$

Exercise 1.2

(a)

We have to put the ray into a plane:

$$
\begin{aligned}
& (p-a) \cdot n = 0 \\
\Leftrightarrow \quad & (r(t)-a) \cdot n = 0 \\
\Leftrightarrow \quad & (o_x + td_x - ax) \cdot n_x + (o_y + td_y - a_y) \cdot n_y + (o_z + td_z - a_z) \cdot n_z = 0
\end{aligned}
$$

Now, there is a case distinction.

1. the ray is parallel to the plane and also the ray is in the plane
   $\Rightarrow d \cdot n = 0$ and $(o-a) \cdot n = 0$
   $\Rightarrow$ for all t, the equation is satisfied ✓
2. the ray is parallel to the plane and the ray is not in the plane
   $\Rightarrow d \cdot n = 0$, but $(o-a) \cdot n \neq 0$
   $\Rightarrow$ no t satisfies the equation ✓
3. the ray is not parallel to the plane (and obviously the ray is not in the plane)
   $\Rightarrow$ exactly one intersection with

   $$t = \frac{(a_x - o_x) \cdot n_x + (a_y - o_y) \cdot n_y + (a_z - o_z) \cdot n_z}{d_x \cdot n_x + d_y \cdot n_y + d_z \cdot n_z} = \frac{(a-o) \cdot n}{d \cdot n}$$

   Since d not perpendicular to n, $d \cdot n \neq 0$ ✓

(b)

Now we put the ray into a sphere:

$$
\begin{aligned}
& (p-C)^2 - R^2 = 0 \\
\Leftrightarrow\ & (p_x - C_x)^2 + (p_y - C_y)^2 + (p_z - C_z)^2 - R^2 = 0 \\
\Leftrightarrow\ & (o_x + td_x - C_x)^2 + (o_y + td_y - C_y)^2 + (o_z + td_z - C_z)^2 - R^2 = 0 \\
\Leftrightarrow\ & (td_x + (o_x - C_x))^2 + (td_y + (o_y - C_y))^2 + (td_z - (o_z - C_z))^2 - R^2 = 0 \\
\Leftrightarrow\ & t^2 d_x^2 + 2td_x(o_x - C_x) + (o_x - C_x)^2 \\
& + t^2 d_y^2 + 2td_y(o_y - C_y) + (o_y - C_y)^2 \\
& + t^2 d_x^2 + 2td_x(o_x - C_x) + (o_x - C_x)^2 - R^2 = 0 \\
\Leftrightarrow\ & t^2(d_x^2 + d_y^2 + d_z^2) + 2t(d_x(o_x - C_x) + d_y(o_y - C_y) + d_z(o_z - C_z)) \\
& + (o_x - C_x)^2 + (o_y - C_y)^2 + (o_z - C_z)^2 - R^2 = 0 \\
\Leftrightarrow\ & t^2 \cdot d^2 + 2t(d(o-C)) + (o-C)^2 - R^2 = 0 \\
\Rightarrow\ & t = \frac{-2(d(o-C)) \pm \sqrt{4(d(o-C))^2 - 4d^2 \cdot (o - C^2) - R^2}}{2d^2} \\
\Rightarrow\ & t = \frac{-d(o-C) \pm \sqrt{M}}{d^2} \quad \text{where } M = (d(o-C))^2 - d^2(o-C)^2 + R^2
\end{aligned}
$$

There are three possible cases:

1. $M = 0 \wedge t_1 > 0 \wedge t_2 > 0$: there exists exactly one solution.
   The ray is tangential. ✓
2. $M < 0 \wedge t_1 < 0 \wedge t_2 < 0$: there exists no solution.
   The ray is missing the sphere. ✓
3. $M > 0 \wedge t_1 > 0 \wedge t_2 > 0$: there exist two solutions.
   The ray hits the sphere. ✓

Exercise 1.4

(a)

We have a ray $R(t) = O + t \cdot D$ and a quadric $ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0$. Now we can start with the following equation:

$$
\begin{aligned}
& a(o_x + td_x)^2 + b(o_y + td_y)^2 + c(o_z + td_z)^2 + d(o_x + td_x)(o_y + td_y) \\
& + e(o_x + td_x)(o_z + td_z) + f(o_y + td_y)(o_z + td_z) + g(o_x + td_x) + h(o_y + td_y) \\
& + i(o_z + td_z) + j = 0 \\
\Leftrightarrow\ & a(o_x^2 + 2o_x td_x + t^2 d_x^2) + b(o_y^2 + 2o_y td_y + t^2 d_y^2) + c(o_z^2 + 2o_z td_z + t^2 d_z^2) \\
& + do_x o_y + do_x td_y + dtd_x o_y + dtd_x td_y + eo_x o_z + eo_x td_z + etd_x o_z + etd_x td_z \\
& + fo_y o_z + fo_y td_z + ftd_y o_z + ftd_y td_z + go_x + gtd_x + ho_y + htd_y \\
& + io_z + otd_z + j = 0 \\
\Leftrightarrow\ & t^2 \cdot (\underbrace{ad_x^2 + bd_y^2 + cd_z^2 + dd_x d_y + ed_x d_z + fd_y d_z}_{\alpha}) \\
& + t \cdot (\underbrace{2ao_x d_x + 2bo_y d_y + 2co_z d_z + do_x d_y + dd_x o_y + eo_x d_z + ed_x o_z}_{\beta} \\
& \quad \underbrace{+ fo_y d_z + fd_y o_z + gd_x + hd_y + id_z}_{\beta}) \\
& + \underbrace{ao_x^2 + bo_y^2 + co_z^2 + do_x o_y + eo_x o_z + fo_y o_z + go_x + ho_y + io_z + j}_{\gamma} = 0 \\
\Leftrightarrow\ & \alpha t^2 + \beta t + \gamma = 0 \\
\Rightarrow\ & t_{1,2} = -\frac{\beta}{2\alpha} \pm \sqrt{\frac{\beta^2}{4\alpha^2} - \frac{\gamma}{\alpha}}
\end{aligned}
$$

(b)

We take the formula of the sphere:

$$
\begin{aligned}
& (x - C_x)^2 + (y - C_y)^2 + (z - C_z)^2 = R^2 \\
\Leftrightarrow\ & x^2 - 2xC_x + C_x^2 + y^2 - 2yC_y^2 + C_y^2 + z^2 - 2zC_z + C_z^2 - R^2 = 0 \\
\Leftrightarrow\ & x^2 + y^2 + z^2 - 2xC_x - 2yC_y - 2zC_z + C_x^2 + C_y^2 + C_z^2 - R^2 = 0
\end{aligned}
$$

Now we represent the equation above by using the quadric formula with

$$a = 1, b = 1, c = 1, d = 0, e = 0, f = 0,$$

$$g = -2C_x, h = -2C_y, i = -2C_z, j = C_x^2 + C_y^2 + C_z^2 - R^2$$

$$
\begin{aligned}
\Rightarrow \alpha &= d_x^2 + d_y^2 + d_z^2 = d^2 \\
\beta &= 2o_xd_x + 2o_yd_y + 2o_zd_z - 2C_xd_x - 2C_yd_y - 2C_zd_z \\
&= 2((o - C)d) \\
\gamma &= o_x^2 + o_y^2 + o_z^2 + 2C_xo_x - 2C_yo_y - 2C_zo_z + C_x^2 + C_y^2 + C_z^2 - R^2 \\
&= o_x^2 - 2C_xo_x + C_x^2 + o_y^2 - 2C_yo_y + C_y^2 + o_z^2 - 2C_zo_z + C_z^2 - R^2 \\
&= (o_x - C_x)^2 + (o_y - C_y)^2 + (o_z - C_z)^2 - R^2 \\
&= (o - C)^2 - R^2
\end{aligned}
$$

Now we can state the values for t:

$$
\begin{aligned}
t_{1,2} &= -\frac{2((o - C)d)}{2d^2} \pm \sqrt{\frac{(2((o - C)d)^2}{4(d^2)^2} - \frac{(o - C)^2 - R^2}{d^2}} \\
&= \frac{(o - C)d}{d^2} \pm \sqrt{\frac{((o - C)d)^2}{(d^2)^2} - \frac{(o - C)^2 - R^2}{d^2}} \\
&= \frac{(o - C)d \pm \sqrt{((o - C)d)^2 - d^2(o - C)^2 - R^2}}{d^2}
\end{aligned}
$$