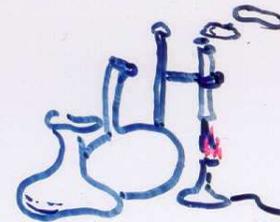THE UNIFICATION LABORATORY

EXPERIMENT 1:

GIVEN:

$$U1: \quad \frac{f(g(a,y),x)}{f(x,g(a,z))}$$

SOLUTION:

$$\sigma_1 = \{x \mapsto g(a,y), z \mapsto y\}$$
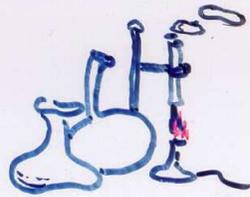
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RIMENT **1**:

N:

$$U1: \frac{f(g(a,Y),x)}{f(x,g(a,z))}$$

SOLUTION:

$$\sigma_1 = \{x \mapsto g(a,Y), z \mapsto Y\}$$

EXPERIMENT **2**:

GIVEN: U1 as before
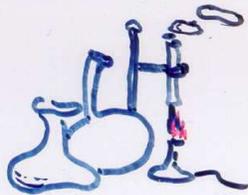
AND C: $f(x,Y) = f(Y,x)$

SOLUTION:

$$\sigma_1 \text{ AND } \sigma_2 = \{z \mapsto Y\}$$

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...IFICATION LABORATORY

...RIMENT 1:

...:
U1:
$$\frac{f(g(a,y),x)}{f(\;x\;,g(a,z))}$$

SOLUTION:
$$G_1 = \{x \mapsto g(a,y)\,,\,z \mapsto y\}$$

EXPERIMENT 3:

GIVEN: U1 as before

AND $A: f(x,f(y,z)) = f(f(x,y),z)$

SOLUTION:

$$G_1 \text{ AND } G_3 = \{x \mapsto f(g(a,y),g(a,y))\,,\,z \mapsto y\}$$

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ᵢᶠⁱCATION LABORATORY

EXPERIMENT 1:

U1:
$$\frac{f(g(a,y),x)}{f(x,g(a,z))}$$
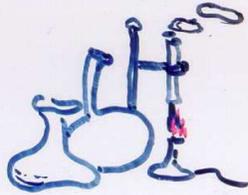
SOLUTION:
$$G_1 = \{x \mapsto g(a,y), z \mapsto y\}$$

EXPERIMENT 2:

GIVEN: U1 as before

AND C: $f(x,y) = f(y,x)$

AND A: $f(x,f(y,z)) = f(f(x,y),z)$

SOLUTION:

THEORY UNIFICATION

- TERMINOLOGY
- E- UNIFICATION:
  - C- UNIFICATION
    $$f(x,y) = f(y,x)$$
  - A- UNIFICATION
    $$f(x, f(y,z)) = f(f(x,y),z)$$
  - AC- UNIFICATION

  T- UNIFICATION

■ **Birkhoff's Calculus for Equational Logic**

▶ $\mathbb{E} \vdash t = t$

▶ $\mathbb{E} \vdash s = t$, if $E \vdash t = s$

▶ $\mathbb{E} \vdash r = t$, if $\mathbb{E} \vdash r = s$ and $\mathbb{E} \vdash s = t$

▶ $\mathbb{E} \vdash f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)$, if $\mathbb{E} \vdash s_1 = t_1, \ldots, s_n = t_n$

▶ $\mathbb{E} \vdash \sigma s = \sigma t$, if $\mathbb{E} \vdash s = t$ and $\sigma \in \Sigma$

With $=_\mathbb{E}$ we denote the smallest relation on $\mathcal{T}_\Sigma$, that is closed under the upper rules (the smallest substitution invariant congruence relation).

- Consider T as additional axioms extensions of $\mathbb{E}$ (i.e., T = {} is standard unification)

- Theory Unification Problem: $\langle s = t \rangle_T$:

$$\boxed{\exists \sigma \in \Sigma . \sigma s =_T \sigma t \text{ for } T \in \mathcal{T}_=}$$

- Theory Matching Problem: $\langle s \leq t \rangle_T$:

$$\boxed{\exists \mu \in \Sigma . \mu s =_T t \text{ for } T \in \mathcal{T}_=}$$

- For $\langle s = t \rangle_T$
  - ► Σ: set of substitutions
  - ► $\mathbb{U}_T(s = t)$: set of unifiers
  - ► $\mu\mathbb{U}_T(s = t)$: minimal set of unifiers
- **Correctness**: $\forall \sigma \in \mu\mathbb{U}_T(s = t).\sigma s =_T \sigma t$
- **Completeness**: $\forall \delta \in \mathbb{U}_T(s = t).\exists \sigma \in \mu\mathbb{U}_T(s = t)$ such that $\sigma \sqsubseteq \delta$
- **Minimality**: $\forall \tau, \sigma \in \mu\Sigma.\sigma \not\sqsubseteq \tau$

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- For $\langle s = t \rangle_T$

  - ▶ Σ: set of substitutions

  - ▶ $\mathbb{U}_T(s = t)$: set of unifiers modulo theory T

  - ▶ $\mu\mathbb{U}_T(s = t)$: minimal set of unifiers modulo theory T

- Typehierarchy:

(i)	$T \in \mathcal{U}_1$	is **unitary** if	$\lvert \mu\mathbb{U}_T \rvert \leq 1$
(ii)	$T \in \mathcal{U}_\omega$	is **finitary** if	$\lvert \mu\mathbb{U}_T \rvert \in \mathbb{N}$
(iii)	$T \in \mathcal{U}_\infty$	is **infinitary** if	$\exists \langle s = t \rangle_T$ such that $\lvert \mu\mathbb{U}_T \rvert \in \infty$
(iv)	$T \in \mathcal{U}_0$	**nullary** **type zero**	otherwise

1. $\mu\mathbb{U} \subseteq \mathbb{U}$

2. $\forall \delta \in \mathbb{U} \, \exists \sigma \in \mu\mathbb{U} : \; \delta \geq \sigma$

3. $\forall \sigma, \tau \in \mu\mathbb{U} : \; \sigma \geq \tau \; \Rightarrow \; \sigma = \tau$

Theorem: The basis $\mu\mathbb{U}$ is unique

- Theory T formulated as set of equation E

- E-Unification-Problem Γ:

$$\langle s_i = t_i : 1 \leq i \leq n \rangle_E$$

- E-Unification-Sets:

$$\mathbb{U}_E(\Gamma) = \{\sigma \in \Sigma . \sigma s_i =_E \sigma t_i,\ 1 \leq i \leq n\}$$

Example: $\Gamma = \{f(g(a,y),x) = f(x,g(a,z))\}$

$$\mathbb{U}_{\{f(x,y)=f(y,x)\}}(\Gamma) := \{x \leftarrow g(a,y), y \leftarrow z\}, \qquad \{y \leftarrow z\}\}$$

- E-Equal on V: $\sigma =_E \tau[V] \Leftrightarrow \sigma x =_E \tau x, \forall x \in V$

- E-Instance on V: $\sigma \geq_E \tau[V] \Leftrightarrow \exists \rho . \sigma =_E \rho \circ \tau[V]$

- E-Equivalent on V: $\sigma \equiv_E \tau[V] \Leftrightarrow \sigma \geq_E \tau[V]\ \&\ \tau \geq_E \sigma[V]$

■ **Lemma:**

$\forall \sigma, \tau : \tau \geq_E \sigma [V]$ if $\sigma \in \mathbb{U}_E(\Gamma)$ then $\tau \in \mathbb{U}_E(\Gamma)$

■ **Theorem:**

1. $\geq_E [V]$ is a quasi-ordering
2. $\equiv_E [V]$ is an equivalence relation

(1) $\mu\mathbb{U}_E(\Gamma) \subseteq \mathbb{U}_E(\Gamma)$ (Correctness)

(2) $\forall \delta \in \mathbb{U}_E(\Gamma) \; \exists \sigma \in \mu\mathbb{U}_E(\Gamma) : \; \delta \geq_E \sigma \, [V]$ (Completeness)

(3) $\forall \sigma, \delta \in \mu\mathbb{U}_E(\Gamma) : \; \sigma \geq_E \delta \, [V] \; \Rightarrow \; \sigma \equiv_E \delta$ (Minimality)

Corrollary:

1. The set of most general E-Unifiers is unique (modulo E-Equivalence).

2. The sets of MGUs have same cardinality.

3. Replacement of elements in $\mu\mathbb{U}$ by E-Equivalent ones gives again a basis.

Theory	Type	Unification Decidable?	$\mu\mathbb{U}_T$?
$\{\,\}$	1	Yes	Yes
A	∞	Yes	Yes
C	ω	Yes	Yes
I	ω	Yes	Yes
A+C	ω	Yes	Yes
A+I	0	Yes	No
C+I	ω	Yes	Yes
A+C+I	ω	Yes	Yes

Theory	Type	Unification Decidable?	$\mu\mathbb{U}_T$?
D	∞	?	Yes
D+A	∞	No	Yes
D+C	∞	?	Yes
D+A+C	∞	No	Yes
D+A+I	?	Yes	No
Hom	1	Yes	Yes
Hom+A	∞	No	Yes
Applications			

Summary

- You can make unification faster using special procedures and special representations

- You can unify modulo a theory, but pay a price (decidable/undecidable, more/infinitely many unifiers)

- Unification theory develops means to study the decidability and (minimal) solution sets.

$\Longrightarrow$ **Prof. Siekmann is looking for a student (Hiwi) to implement a framework to interactively study new theories for unification**

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