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Fachrichtung 6.2 – Informatik
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5th Theoretical Assignment in Artificial Intelligence (SS 2005)
Solutions

Exercise 5.1 **(10 P)**

In the following, we specify a set of first order logic formulas:

(A) $\forall x. \neg(x < x)$

(B) $\forall x, y, z. (x < y \wedge y < z) \Rightarrow x < z$

(C) $\forall x. x < S(x)$

(D) $N(0)$

(E) $\forall x. N(x) \Rightarrow N(S(x))$

A model for this first-order language is given by a nonempty set $\mathcal{D}$ of objects, a binary relation $|<|$ on $\mathcal{D}$ (to interpret $<$), a function $|S|$ from $\mathcal{D}$ to $\mathcal{D}$ (to interpret S), a subset $|N|$ of $\mathcal{D}$ (to interpret N) and an element $|0|$ of $\mathcal{D}$ (to interpret 0).

1. Give a model which satisfies the set of formulas {**(A)**,**(C)**,**(D)**,**(E)**} and such that $\mathcal{D}$ contains exactly two objects. (5 P)

2. Give a model that satisfies the set of formulas {**(A)**,**(B)**,**(C)**,**(D)**,**(E)**}. (Hint: $\mathcal{D}$ can be infinite.) (5 P)

For each part, justify your answer!

Solution:

1. *Let $\mathcal{D}$ be $\{0, 1\}$, $|<|$ be $\{(0, 1), (1, 0)\}$, $|S|(0) = 1$, $|S|(1) = 0$, $|N| = D$ and $|0| = 0$.* **(A)** *is true since $(0 < 0)$ and $(1 < 1)$ are false.* **(C)** *is true since $(0 < 1)$ and $(1 < 0)$ are true.* **(D)** *and* **(E)** *are true since $|N| = D$. (Note that* **(B)** *is false since $(0 < 1)$ and $(1 < 0)$ are true, but $(0 < 0)$ is false.)*

2. *Let $\mathcal{D}$ be the natural numbers: $\{0, 1, 2, \ldots\}$, $|<|$ be the "strictly-less-than" relation, $|S|$ be the successor function ($|S|(n) = n + 1$), $|N|$ be $\mathcal{D}$, and $|0|$ be 0. (A) is true since no natural number is less than itself. (B) is true since the strictly-less-than relation is transitive. (C) is true since we always have $n < n + 1$. (D) and (E) are true since $|N|$ is $\mathcal{D}$.*

   Note that (D) would also hold if we choose $|0|$ to be, say, 527. Also, (D) and (E) would hold if we choose $|0|$ to be 527 and choose $|N|$ to be $\{527, 528, 529, \ldots\}$.

Exercise 5.2 **(13 P)**

1. Represent the following statement in first order logic.

   - Every student attends some class.

   (5 P)

2. Convert your first-order statement above into clause normal form as described in class. In this example, you should perform the following steps:

   (a) Eliminate implications in favor of disjunction and negation.

   (b) Skolemize to remove existential quantifiers.

   (c) Remove universal quantifiers.

   (d) Distribute $\wedge$ over $\vee$.

   Show the result of each step! (8 P)

Solution:

1. $\forall x. Stud(x) \Rightarrow \exists y. Attends(x, y) \wedge Class(y)$

2. *(a) Eliminate Implication:* $\forall x. \neg Stud(x) \vee \exists y. Attends(x, y) \wedge Class(y)$

(b) Skolemize: $\forall x. \neg Stud(x) \vee Attends(x, f(x)) \wedge Class(f(x))$

(c) Remove $\forall$*:* $\neg Stud(x) \vee Attends(x, f(x)) \wedge Class(f(x))$

(d) Distribute $\wedge$ *over* $\vee$*:*

$$(\neg Stud(x) \vee Attends(x, f(x))) \wedge (\neg Stud(x) \vee Class(f(x)))$$

Exercise 5.3 **(10 P)**

Identify the most general unifier (if one exists) for the following pairs of formulas. Constants are illustrated as capitals, variables as lower case letters.

1. $Older(Father(y), y)$, $Older(Father(x), John)$ (5 P)
2. $Q(y, G(A, B))$, $Q(G(x, x), y)$ (5 P)

Solution:

1. $\{y/John, x/John\}$

2. no unifier.

Exercise 5.4 **(20 P)**

Let S be the set of the following four clauses:

(A) $P(A, A)$

(B) $\neg P(B, B)$

(C) $\neg P(x,x) \vee P(x, f(x,y))$

(D) $\neg P(x, f(x,y)) \vee P(y,y)$

The set S is inconsistent.

1. In general, the Herbrand Universe of a set of clauses is the set of all ground terms (terms with no variables) that can be constructed from constants and function symbols which occur in the set of clauses. Describe the Herbrand Universe H_S of S. (5 P)

2. Give all ten (10) ground instances of the clauses **(A)**, **(B)**, **(C)** and **(D)** the variables to range over the two terms A and B. (5 P)

3. Give a set of four of the ten (10) ground clauses above which is inconsistent. Also, give a refutation (i.e., derive the empty clause) using resolution and factoring. Let Ψ be the set of all ground clauses which appear in this refutation. (5 P)

4. Find a refutation of the clauses in S (with variables) using resolution and factoring with the following property: Every clause involved in this refutation has a ground instance in Ψ. (This is the process of "lifting" the ground refutation to a refutation with variables.) (5 P)

Solution:

1. *H_S is $\{A, B, f(A,A), f(A,B), f(B,A), f(B,B), f(f(A,A),A), \ldots\}$. That is, H_S is the set of all terms built using f starting from A and B.*

2. *Ten ground clauses:*
   - $P(A,A)$
   - $\neg P(B,B)$
   - $\neg P(A,A) \vee P(A, f(A,A))$
   - $\neg P(A,A) \vee P(A, f(A,B))$
   - $\neg P(B,B) \vee P(B, f(B,A))$
   - $\neg P(B,B) \vee P(B, f(B,B))$
   - $\neg P(A, f(A,A)) \vee P(A,A)$
   - $\neg P(A, f(A,B)) \vee P(B,B)$
   - $\neg P(B, f(B,A)) \vee P(A,A)$
   - $\neg P(B, f(B,B)) \vee P(B,B)$

3. *Four inconsistent ground clauses:*
   - $P(A,A)$
   - $\neg P(B,B)$
   - $\neg P(A,A) \vee P(A, f(A,B))$
   - $\neg P(A, f(A,B)) \vee P(B,B)$

   Resolution Refutation:

   - $P(A, f(A,B))$ *follows by resolving* $P(A,A)$ *with* $\neg P(A,A) \vee P(A, f(A,B))$.

- *$P(B,B)$ follows by resolving $P(A, f(A,B))$ with $\neg P(A, f(A,B)) \vee P(B,B)$.*
- *Empty clause follows by resolving $P(B,B)$ with $\neg P(B,B)$.*

4. *Lifted Resolution Refutation:*

   - *$P(A, f(A,y))$ follows by resolving $P(A,A)$ with $\neg P(x,x) \vee P(x, f(x,y))$. (Note that $P(A, f(A,B))$ is an instance of $P(A, f(A,y))$.)*
   - *$P(y,y)$ follows by resolving $P(A, f(A,z))$ (renamed y to z) with $\neg P(x, f(x,y)) \vee P(y,y)$. (Note that $P(B,B)$ is an instance of $P(y,y)$.)*
   - *Empty clause follows by resolving $P(y,y)$ with $\neg P(B,B)$. (Note that the empty clause is an instance of the empty clause.)*

Exercise 5.5 **(6 P)**

The *unit preference strategy* only allows resolution steps if one of the clauses is a unit clause (that is, a clause with one literal). Let S be the set of the following four propositional clauses:

(A) $P \vee Q$

(B) $\neg P \vee Q$

(C) $P \vee \neg Q$

(D) $\neg P \vee \neg Q$

Explain why the empty clause cannot be derived from S using the unit preference strategy. (Your explanation should not be more than three sentences.)

Solution:

Since none of the clauses is a unit clause, **the unit preference strategy does not allow any resolution step** *using the clauses.* **Factoring cannot be applied to any of the clauses.** *Therefore, no clause (including the empty clause) can be derived from the four clauses if we use the unit preference strategy.*

Exercise 5.6 **(15 P)**

The *set-of-support (sos) strategy* only allows inference rules (resolution or factoring) to be applied when one of the given clauses is in the set-of-support (sos). Whenever a new clause is inferred using resolution or factoring, the new clause is included in the set-of-support. Let Φ be the set of four clauses

(A) $P(0)$

(B) $\neg P(S(S(S(0))))$

(C) $\neg P(x) \vee P(S(x))$

(D) $\neg P(S(x)) \vee P(x)$

1. Assume the set-of-support is initially $\{(A)\}$. Derive the empty clause from Φ using the set-of-support strategy. (5 P)

2. Assume the set-of-support is initially $\{(B)\}$. Derive the empty clause from Φ using the set-of-support strategy. (5 P)

3. Assume the set-of-support is initially $\{(D)\}$. Give four clauses that can be derived from Φ using the set-of-support strategy. (5 P)

Solution:

1.
- $P(S(0))$ *- resolution with* **(A)** *and* **(C)**
- $P(S(S(0)))$ *- resolution with previous and* **(C)**
- $P(S(S(S(0))))$ *- resolution with previous and* **(C)**
- *empty clause - resolution with previous and* **(B)**

2.
- $\neg P(S(S(0)))$ *- resolution with* **(B)** *and* **(C)**
- $\neg P(S(0))$ *- resolution with previous and* **(C)**
- $\neg P(0)$ *- resolution with previous and* **(C)**
- *empty clause - resolution with previous and* **(A)**

3. The first four clauses we can derive with **(D)** *as the initial sos:*

- $\neg P(S(S(S(S(0)))))$ *- resolution with* **(D)** *and* **(B)**
- $\neg P(x) \vee P(x)$ *- resolution with* **(D)** *and* **(C)**
- $\neg P(S(x)) \vee P(S(x))$ *- resolution with* **(D)** *and* **(C)** *(with different choices of literals)*
- $\neg P(S(S(x))) \vee P(x)$ *- resolution with* **(D)** *and* **(D)**

One can continue and also derive

- $\neg P(S(S(S(S(S(0))))))$, $\neg P(S(S(S(S(S(S(0)))))))$, ...
- $\neg P(S(S(S(x)))) \vee P(x)$, $\neg P(S(S(S(S(x))))) \vee P(x)$, ...

Exercise 5.7 **(20 P)**

Given the following clause forms, prove whether there exists an $x \in \{John, Mike, Tom\}$ that satisfies both $Climber(x)$ and $\neg Skier(x)$? If it exists, who is x:

1. $Alp(Tom)$
2. $Alp(Mike)$
3. $Alp(John)$
4. $\neg Alp(x) \vee Skier(x) \vee Climber(x)$
5. $\neg Skier(x) \vee Like(x, Snow)$
6. $\neg Climber(x) \vee \neg Like(x, Rain)$
7. $\neg Like(Tom, y) \vee \neg Like(Mike, y)$
8. $Like(Tom, y) \vee Like(Mike, y)$
9. $Like(Tom, Rain)$
10. $Like(Tom, Snow)$

Solution:

Mike is the climber who is not a skier.

- $\neg Like(Mike, Snow)$ by resolving $Like(Tom, Snow)$ and $\neg Like(Tom, y) \vee \neg Like(Mike, y)$ where $\{y/Snow\}$
- $\neg Skier(Mike)$ by resolving previous and $Like(x, Snow) \vee \neg Skier(x)$ where $\{x/Mike\}$
- $Skier(Mike) \vee Climber(Mike)$ by resolving $Alp(Mike)$ and $\neg Alp(x) \vee Skier(x) \vee Climber(x)$ where $\{x/Mike\}$
- $Climber(Mike)$ by resolving $\neg Skier(Mike)$ and $Skier(Mike) \vee Climber(Mike)$

Note: *Another solution involves starting with a clause*

$$\neg Climber(x) \vee Skier(x)$$

representing the negation of the conclusion. Similar resolution steps are involved, and in the end, one can get a contradiction using only the ground instance

$$\neg Climber(Mike) \vee Skier(Mike)$$

The next problem should make it clear that, in a general problem, we can prove an x exists satisfying a property though no single, particular x can be proven to have the property.

Exercise 5.8 **(6 P)**

Suppose we are given the clause

(GM) $GoodMan(John) \vee GoodMan(Mike) \vee GoodMan(Tom)$

1. Using clause **(GM)**, prove $\exists x GoodMan(x)$ by refuting $\neg \exists x GoodMan(x)$ using resolution and factoring. (3 P)
2. Is it possible to find a particular $x \in \{John, Mike, Tom\}$ such that $GoodMan(x)$ follows from clause **(GM)**? (3 P)

Solution:

1. *The clause normal form of $\neg \exists x GoodMan(x)$ is $\neg GoodMan(x)$ (where x is a variable). We obtain a refutation as follows:*
   - $GoodMan(Mike) \vee GoodMan(Tom)$ *by resolution of* $\neg GoodMan(x)$ *with* **(GM)** *(where* $\{x/John\}$*).*
   - $GoodMan(Tom)$ *by resolution of* $\neg GoodMan(x)$ *with previous (where* $\{x/Mike\}$*)*
   - *Empty clause by resolution of* $\neg GoodMan(x)$ *with previous (where* $\{x/Tom\}$*)*
2. *No. For each* $Guy \in \{John, Mike, Tom\}$*, it is consistent to assume* $\neg GoodMan(Guy)$ *along with* **(GM)***.*

Bonus Exercise 5.9 **(12 P)**

Let Φ be the set with the two clauses $(0 = 1)$ and $\neg(1 = 0)$ (where 0 and 1 are constant symbols). Either $(0 = 1)$ or $\neg(1 = 0)$ must be false in any first-order model with equality. (In this sense, the set Φ is *unsatisfiable*).

1. Explain why it is *not possible* to derive the empty clause from Φ using the resolution and factoring rules. (Your explanation should not be more than three sentences.) (3 P)

2. Explain why the unsatisfiable clause set Φ does not provide a counterexample to the completeness theorem for resolution given in class. (Your explanation should not be more than two sentences.) (3 P)

3. Derive the empty clause from $\Phi \cup \{x = x\}$ using the resolution, factoring and paramodulation rules. (3 P)

4. Is it possible to derive the empty clause from Φ using the resolution, factoring and paramodulation rules? (Answering "Yes" or "No" is sufficient.) (3 P)

Solution:

1. *Resolution cannot be applied since* $(0 = 1)$ *and* $(1 = 0)$ *do not unify (there are no variables). Factoring cannot be applied as every clause is a unit clause (i.e., has only one literal). Thus no clause (including the empty clause) can be derived from* Φ *using resolution and factoring.*

2. *The (refutation) completeness theorem for resolution was for first-order logic without equality!*

3.
   - $\neg(0 = 0)$ *by paramodulation with* $(0 = 1)$ *and* $\neg(1 = 0)$
   - *empty clause by resolution of previous with* $(x = x)$.

4. **No.** *(We can derive* $\neg(0 = 0)$ *and* $\neg(1 = 1)$ *by paramodulation, but there is no way to obtain the empty clause without assuming some reflexivity clause.)*
