Introduction to Computational Logic, SS 2006: Solution for assignment 12

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Exercise 12.1 (Boolean Quantifiers)

$\forall f = f0 \wedge f1$	
$\forall f \leftrightarrow f0 \wedge f1$	Eq

By D↔, And it suffices to show the implication in each direction separately.

$\rightarrow$: $\forall f \rightarrow f0 \wedge f1$	
$\forall f \wedge f0 \wedge f1 \rightarrow f0 \wedge f1$	$\forall I$, GR
1	Taut

$\leftarrow$: $f0 \wedge f1 \rightarrow \forall f$		
$f0 \wedge f1 \rightarrow fx$		Pull $\rightarrow$, Gen$\forall$
$f0 \wedge f1 \rightarrow f0$	$f0 \wedge f1 \rightarrow f1$	BCA (on x)
1	1	Taut

Exercise 12.2 (Cantor)

$\overline{\exists f \forall g \exists x \forall h. h(fx) \leftrightarrow hg}$	
$\forall f \exists g \forall x \exists h. h(fx) \leftrightarrow \overline{hg}$	dM, Taut
$\forall x. \exists h. h(fx) \leftrightarrow \overline{h(\lambda x. \overline{fxx})}$	Gen$\forall$, Gen$\exists$($g := \lambda x. \overline{fxx}$)
$(\lambda g. gx)(fx) \leftrightarrow \overline{(\lambda g. gx)(\lambda x. \overline{fxx})}$	Gen$\forall$, Gen$\exists$($h := \lambda g. gx$)
$fxx \leftrightarrow \overline{\overline{fxx}}$	β
1	Taut

Exercise 12.3 (Strange)

a) $\forall_{((S\to B)\to B)\to T\to B} f.\forall_T (f\forall_S)$

b) We show $\overline{\forall f.\forall(f\forall)} = 1$ with a backward proof.

$\overline{\forall f.\forall(f\forall)}$	
$\exists f.\overline{\forall(f\forall)}$	dM
$\overline{\forall((\lambda gx.0)\forall)}$	Gen$\exists(f := \lambda gx.0)$
$\overline{\forall(\lambda x.0)}$	β
$\overline{\forall(\lambda x.0) \wedge (\lambda x.0)0}$	$\forall I$, GR
$\overline{\forall(\lambda x.0) \wedge 0}$	β
1	Taut

Here is an alternative conversion proof for $QL \vdash \forall f.\forall(f\forall) = 0$

$\forall f.\forall(f\forall)$	
$=\forall f.\forall(f\forall) \wedge (\lambda f.\forall(f\forall))(\lambda xy.0)$	$\forall I$, GR
$=\forall f.\forall(f\forall) \wedge \forall(\lambda y.0)$	$\beta(2x)$
$=\forall f.\forall(f\forall) \wedge \forall(\lambda y.0) \wedge (\lambda y.0)0$	$\forall I$, GR
$=\forall f.\forall(f\forall) \wedge \forall(\lambda y.0) \wedge 0$	β
$=0$	Taut

Exercise 12.4 (Tricky)

$\exists x.fxy \to fzx$	
$(\forall x.fxy) \to (\exists x.fzx)$	Pull $\to$
$\exists x \exists x'.fxy \to fzx'$	Pull $\to$
$fzy \to fzy$	Gen$\exists(x := z,\ x' := y)$
1	Taut

Here is an alternative conversion proof.

$$
\begin{aligned}
&= \exists x. fxy \to fzx \\
&= \exists x. \overline{fxy} \vee fzx && \text{Taut} \\
&= (\exists x. \overline{fxy}) \vee (\exists x. fzx) && \exists\vee' \\
&= (\exists x. \overline{fxy}) \vee (\lambda x. \overline{fxy})z \vee (\exists x. fzx) \vee (\lambda x. fzx)y = && \exists I, \text{GR } (2x) \\
&= (\exists x. \overline{fxy}) \vee \overline{fzy} \vee (\exists x. fzx) \vee fzy && \beta \ (2x) \\
&= 1 && \text{Taut}
\end{aligned}
$$

Exercise 12.5 (Challenge)

$$
\begin{aligned}
&\exists g \forall x \exists h. h(fx) \wedge \overline{hg} \\
&\exists h. h(fx) \wedge \overline{h(\lambda x. \overline{fxx})} && \text{Gen}\exists, \text{Gen}\forall \\
&\exists h. h(fx) \wedge \overline{h(\lambda x. \overline{fxx})} \vee \exists h. h(fx) \wedge \overline{h(\lambda x. \overline{fxx})} && \text{Taut} \\
&\exists h_1 \exists h_2. h_1(fx) \wedge \overline{h_1(\lambda x. \overline{fxx})} \vee h_2(fx) \wedge \overline{h_2(\lambda x. \overline{fxx})} && \exists\vee \\
&fxx \wedge \overline{\overline{fxx}} \vee \overline{fxx} \wedge \overline{\overline{\overline{fxx}}} && \text{Gen}\exists, \beta \\
&1 && \text{Taut}
\end{aligned}
$$

The used instances of Gen∃ are

$$
\begin{aligned}
g &:= \lambda x. \overline{fxx} \\
h_1 &:= \lambda g. gx \\
h_2 &:= \lambda g. \overline{gx}
\end{aligned}
$$

Alternatively, you can reduce the proof to Exercise 12.2.

$$
\begin{aligned}
&\exists g \forall x \exists h. h(fx) \wedge \overline{hg} \\
&\exists g \forall x \exists h. \overline{h(fx) \to hg} && \text{Taut} \\
&\exists g \forall x \overline{\forall h. h(fx) \to hg} && \text{dM} \\
&\exists g \forall x \overline{\forall h. h(fx) \leftrightarrow hg} && \text{Ex. 12.7} \\
&\exists g \forall x \exists h. h(fx) \leftrightarrow \overline{hg} && \text{dM, Taut}
\end{aligned}
$$

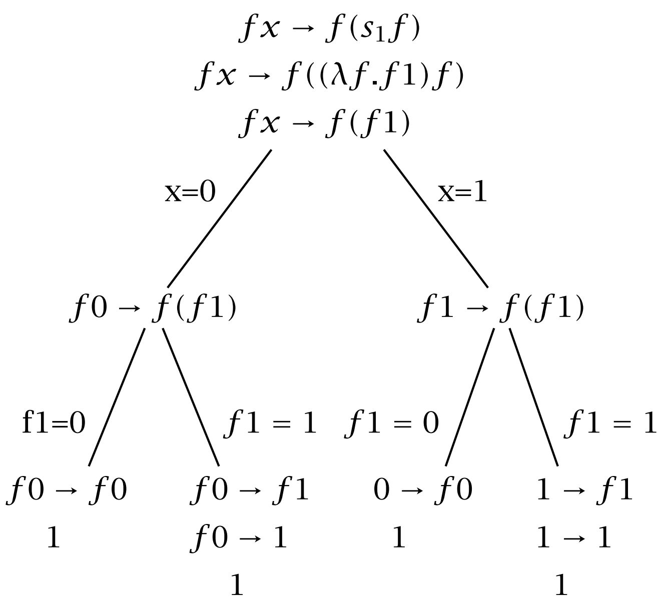
Exercise 12.6 (Boolean Choice)

a) g is a choice function for $\mathbb{B}$ if and only if $g(\lambda x.x) = 1$ and $g(\neg) = 0$. There exist 4 different functions with this property.

b)

$$s_1 = \lambda f.f1$$
$$s_2 = \lambda f.\overline{f0}$$
$$s_3 = \lambda f.f1 \vee \overline{f0}$$
$$s_4 = \lambda f.f1 \wedge \overline{f0}$$

c)



d)

$$\begin{aligned} \exists f &= f(Cf) && D\exists \\ &= f((\lambda f.f1)f) && \text{choose } s_1 \text{ as } C \\ &= f(f1) && \beta \end{aligned}$$

Exercise 12.7 (Definition of Identities)

a)

$(\forall f. fx \to fy) \leftrightarrow (\forall f. fx \leftrightarrow fy)$

By Taut and And we reduce $\leftrightarrow$ to $\leftarrow$ and $\to$

$\leftarrow$:	$(\forall f. fx \to fy) \leftarrow (\forall f. fx \leftrightarrow fy)$	
	$(\forall f. fx \leftrightarrow fy) \to (fx \to fy)$	Pull $\to$, Gen$\forall$
	$\exists f'.(f'x \leftrightarrow f'y) \to fx \to fy$	Pull $\to$
	$(fx \leftrightarrow fy) \to fx \to fy$	Gen$\exists(f' := f)$
	1	Taut

$\to$:	$(\forall f. fx \to fy) \to (\forall f. fx \leftrightarrow fy)$	
	$(\forall f. fx \to fy) \to (fx \leftrightarrow fy)$	Gen$\forall$

Here we again consider 2 cases:

$(\forall f. fx \to fy) \to (fx \to fy)$	
$(fx \to fy) \to fx \to fy$	Pull $\to$, Gen$\exists$
1	Taut

$(\forall f. fx \to fy) \to (fy \to fx)$	
$(\forall f. fx \to fy) \to \overline{fx} \to \overline{fy}$	Taut
$\exists f'.(f'x \to f'y) \to \overline{fx} \to \overline{fy}$	Pull $\to$
$(\overline{fx} \to \overline{fy}) \to \overline{fx} \to \overline{fy}$	Gen$\exists(f' = \lambda x.\overline{fx}), \beta$
1	Taut

There also exists an elegant conversion proof.

$$
\begin{array}{lll}
& \forall f.fx \to fy & \\
= & \forall f(\forall f.fx \to fy) & \forall E \\
= & \forall f(\forall f.fx \to fy) \land (\overline{fx} \to \overline{fy}) & \forall I(f := \lambda x.\overline{fx}), \mathrm{GR}, \beta \\
= & (\forall f.fx \to fy) \land (\forall f.\overline{fx} \to \overline{fy}) & \forall\land \\
= & (\forall f.fx \to fy) \land (\forall f.fy \to fx) & \mathrm{Taut} \\
= & \forall f.(fx \to fy) \land (fy \to fx) & \forall\land' \\
= & \forall f.fx \leftrightarrow fy & \mathrm{Taut}
\end{array}
$$

b) The equation gives you another, more intuitive characterization of identity:

$$x \doteq y = \forall f.fx \leftrightarrow fy$$

With this characterization the symmetrie of $\leftrightarrow$ generalizes to $\doteq$.

Exercise 12.8 (Duality)

$$
\begin{array}{ll}
x \doteq y \leftrightarrow \exists f.fx \not\leftarrow fy & \\
\overline{x \doteq y} \leftrightarrow \exists f.fx \not\leftarrow fy & D \neq \\
\overline{y \doteq x} \leftrightarrow \exists f.fx \not\leftarrow fy & \mathrm{Sym} \\
\overline{\forall f.fy \to fx} \leftrightarrow \exists f.fx \not\leftarrow fy & D \doteq \\
(\exists f.fx \not\leftarrow fy) \leftrightarrow \exists f.fx \not\leftarrow fy & \mathrm{dM} \\
1 & \mathrm{Taut}
\end{array}
$$

Exercise 12.9 (Identity Laws)

a)

$x \doteq y \to y \doteq z \to x \doteq z$	
$(\forall f.fx \to fy) \to (\forall f.fy \to fz) \to (\forall f.fx \to fz)$	$D \doteq$
$(\forall f.fx \to fy) \to (\forall f.fy \to fz) \to fx \to fz$	Pull $\to$, Gen$\forall$
$\exists f'.(\forall f.fx \to fy) \to (f'y \to f'z) \to fx \to fz$	Pull $\to$
$(\forall f.fx \to fy) \to (fy \to fz) \to fx \to fz$	Gen$\exists(f' := f)$
$\exists f'.(f'x \to f'y) \to (fy \to fz) \to fx \to fz$	Pull $\to$
$(fx \to fy) \to (fy \to fz) \to fx \to fz$	Gen$\exists(f' := f)$
1	Taut

b)

$x \doteq y \to fx \doteq fy$	
$(\forall f.fx \to fy) \to (\forall g.g(fx) \to g(fy))$	$D \doteq$
$(\forall f.fx \to fy) \to g(fx) \to g(fy)$	Pull $\to$, Gen$\forall$
$\exists f'.(f'x \to f'y) \to g(fx) \to g(fy)$	Pull $\to$
$(g(fx) \to g(fy)) \to g(fx) \to g(fy)$	Gen$\exists(f' := \lambda x.g(fx)), \beta$
1	Taut

c)

$(\forall x.s \doteq t) \to (\lambda x.s) \doteq (\lambda x.t)$	
$(\forall x.(\lambda x.s)x \doteq (\lambda x.t)x) \to (\lambda x.s) \doteq (\lambda x.t)$	β
$(\lambda x.s) \doteq (\lambda x.t) \to (\lambda x.s) \doteq (\lambda x.t)$	Ext
1	Taut

Exercise 12.10

$\to$:	$x \wedge y \to (\lambda f.fxy) \doteq (\lambda f.f11)$	
	$x \doteq 1 \to y \doteq 1 \to (\lambda f.fxy) \doteq (\lambda f.f11)$	Taut
	$x \doteq 1 \to y \doteq 1 \to (\lambda f.f11) \doteq (\lambda f.f11)$	BRep
	$x \doteq 1 \to y \doteq 1 \to 1$	Ref
	1	Taut

$\leftarrow$:	$(\lambda f.fxy) \doteq (\lambda f.f11) \to x \wedge y$	
	$\exists g.(g(\lambda f.fxy) \to g(\lambda f.f11)) \to x \wedge y$	$D\doteq$, Pull $\to$
	$(\overline{x \wedge y} \to \overline{1 \wedge 1}) \to x \wedge y$	Gen$\exists$($g := \lambda h.h(\lambda xy.\overline{x \wedge y})$)
	1	Taut