# Introduction to Computational Logic, SS 2006: Solution for assignment 11

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**Exercise 11.1 (Inconsistence)** a) Claim: QL $\not\vdash 0 = 1$

Proof: By Contradiction. Let QL $\vdash 0 = 1$. Then $\mathcal{A}0 = \mathcal{A}1$ in every model of QL (by soundness). Contradiction.

b)

| | | |
|---|---|---|
| " $\dashv$ ": | $0 = \bar{0}$ | $x = \bar{x}$ |
| | $= 1$ | Negation |
| " $\vdash$ ": | $x = x \wedge 1$ | Identity |
| | $= x \wedge 0$ | $0 = 1$ |
| | $= 0$ | Dominance |
| | $= \bar{x} \wedge 0$ | Dominance |
| | $= \bar{x} \wedge 1$ | $0 = 1$ |
| | $= \bar{x}$ | Identity |

c)

| | | |
|---|---|---|
| " $\dashv$ ": | $0 = 1$ | $x = y$ |
| " $\vdash$ ": | $x = x \wedge 1$ | Identity |
| | $= x \wedge 0$ | $0 = 1$ |
| | $= 0$ | Dominance |
| | $= y \wedge 0$ | Dominance |
| | $= y \wedge 1$ | $0 = 1$ |
| | $= y$ | Identity |

**Exercise 11.2 (Duality)** $e = (a = 0)$ where $a$ is a parameter.

**Exercise 11.3 (∃ Defined)**

a)

$$\begin{aligned}
\exists x.0 &= \overline{\forall x.\overline{(\lambda x.0)x}} && \exists f = \overline{\forall x.\overline{fx}} \\
&= \overline{\forall x.\bar{0}} && \beta \\
&= \overline{\forall x.1} && \text{Taut} \\
&= \bar{1} && \forall 1 \\
&= 0 && \text{Taut}
\end{aligned}$$

b)

$$\begin{aligned}
fx \to \exists f &= fx \to \overline{\forall x.\overline{fx}} && \exists f = \overline{\forall x.\overline{fx}} \\
&= \forall x.\overline{fx} \to \overline{fx} && \text{Taut} \\
&= \forall x.\overline{fx} \to (\lambda x.\overline{fx})x && \beta \\
&= 1 && \forall I
\end{aligned}$$

c) " $\dashv$ ": follows from a) and b).

" $\vdash$ ": We have to show QL $\vdash \exists f = \overline{\forall x.\overline{fx}}$.

$$\begin{aligned}
\overline{\forall x.\overline{fx}} &= \exists x.\overline{\overline{fx}} && \text{dM} \\
&= \exists x.fx && \text{Taut} \\
&= \exists f && \eta
\end{aligned}$$

**Exercise 11.4 (Pull Laws →)**

a)

$$\begin{aligned}
q \to \exists f &= \bar{q} \vee \exists f && \text{Taut} \\
&= \exists x.\bar{q} \vee fx && \exists \vee \\
&= \exists x.q \to fx && \text{Taut}
\end{aligned}$$

b)

$$\begin{aligned}
q \to \forall f &= \bar{q} \vee \forall f && \text{Taut} \\
&= \forall x.\bar{q} \vee fx && \forall \vee \\
&= \forall x.q \to fx && \text{Taut}
\end{aligned}$$

c)

$$\begin{aligned}
\exists f \to q &= \overline{\exists f} \vee q && \text{Taut} \\
&= (\forall x.\overline{fx}) \vee q && \text{dM} \\
&= \forall x.\overline{fx} \vee q && \forall\vee \\
&= \forall x.fx \to q && \text{Taut}
\end{aligned}$$

d)

$$\begin{aligned}
\forall f \to q &= \overline{\forall f} \vee q && \text{Taut} \\
&= (\exists x.\overline{fx}) \vee q && \text{dM} \\
&= \exists x.\overline{fx} \vee q && \exists\vee \\
&= \exists x.fx \to q && \text{Taut}
\end{aligned}$$

e)

$$\begin{aligned}
\forall f \to \exists g &= \overline{\forall f} \vee \exists g && \text{Taut} \\
&= (\exists x.\overline{fx}) \vee \exists g && \eta, \text{dM} \\
&= \exists x.\overline{fx} \vee gx && \exists\vee', \beta \\
&= \exists x.fx \to gx && \text{Taut}
\end{aligned}$$

**Exercise 11.5 (Backward Proofs)**

a)

$$\begin{aligned}
\forall f \to \exists f &= \exists x.fx \to fx && \text{Pull} \to \\
&= \exists x.1 && \text{Taut} \\
&= 1 && \exists E
\end{aligned}$$

b)

$$\begin{aligned}
\exists x.fx \to \forall f &= \forall f \to \forall f && \text{Pull} \to \\
&= 1 && \text{Taut}
\end{aligned}$$

c)

$$
\begin{aligned}
& \forall f \vee \forall g \to \forall x. fx \vee gx \\
= \; & \forall x. \forall f \vee \forall g \to fx \vee gx && \text{Pull} \to \\
\vdash\!\dashv \; & \forall f \vee \forall g \to fx \vee gx && \text{Gen}\forall \\
= \; & (\forall yz. fy \vee gz) \to fx \vee gx && \forall \vee \; (2x) \\
= \; & \exists yz. fy \vee gz \to fx \vee gx && \text{Pull} \to \; (2x) \\
\dashv \; & fx \vee gx \to fx \vee gx && \text{Gen}\exists, y = x, z = x \\
= \; & 1 && \text{Taut}
\end{aligned}
$$

d)

$$
\begin{aligned}
& \forall f \to \forall x.\, fx \vee gx \\
= \; & \forall x.\, \forall f \to fx \vee gx && \text{Pull} \to \\
= \; & \forall x.\, (\forall f \to fx) \vee gx && \text{Taut} \\
= \; & \forall x. 1 \vee gx && \forall I \\
= \; & \forall x. 1 && \text{Taut} \\
= \; & 1 && \forall 1
\end{aligned}
$$

e)

$$
\begin{aligned}
& (\exists f \to \forall h) \to \forall x.\, fx \to hx \\
\vdash\!\dashv \; & (\exists f \to \forall h) \to fx \to hx && \text{Pull}\to, \text{Gen}\forall \\
= \; & (\forall y \forall z. fy \to hz) \to fx \to hx && \text{Pull} \to \\
= \; & \exists y \exists z. (fy \to hz) \to fx \to hx && \text{Pull} \to \\
\dashv \; & (fx \to hx) \to fx \to hx && \text{Gen}\exists, y = x, z = x \\
= \; & 1 && \text{Taut}
\end{aligned}
$$

f)

$$
\begin{array}{ll}
(\forall x.\, fx \vee hx) \to \forall f \vee \exists h & \\
= \forall y \exists z.(\forall x.\, fx \vee hx) \to fy \vee hz & \forall\vee, \exists\vee, \text{Pull}\to \\
\dashv (\forall x.\, fx \vee hx) \to fy \vee hy & \text{Gen}\forall, \text{Gen}\exists, z = y \\
= \exists x.(fx \vee hx) \to fy \vee hy & \text{Pull}\to \\
\dashv (fy \vee hy) \to fy \vee hy & \text{Gen}\exists, x = y \\
= 1 & \text{Taut}
\end{array}
$$

g)

$$
\begin{array}{ll}
(\exists x \forall y.fxy) \to \forall y \exists x.fxy & \\
(\exists x \forall y.fxy) \to \exists x.fxy & \text{Pull}\to, \text{Gen}\forall \\
\forall x \exists y'.fxy' \to \exists x.fxy & \text{Pull}\to \\
fxy \to \exists x.fxy & \text{Gen}\forall, \text{Gen}\exists, y' = y \\
\exists x'.fxy \to fx'y & \text{Pull}\to \\
fxy \to fxy & \text{Gen}\exists, x' = x \\
1 & \text{Taut}
\end{array}
$$

h)

$$
\begin{array}{ll}
\forall x.\, fx \to \exists f & \\
\forall x \exists x'.\, fx \to fx' & \text{Pull}\to \\
fx \to fx & \text{Gen}\forall, \text{Gen}\exists, x' = x \\
1 & \text{Taut}
\end{array}
$$

i)

$$
\begin{array}{ll}
(\forall x.\, fx \to gx) \to \forall f \to \forall g & \\
(\forall x.\, fx \to gx) \to \forall f \to gx & \text{Pull}\to, \text{Gen}\forall \\
\exists x'.(\forall x.\, fx \to gx) \to fx' \to gx & \text{Pull}\to \\
(\forall x.\, fx \to gx) \to fx \to gx & \text{Gen}\exists, x' = x \\
\exists x'.(fx' \to gx') \to fx \to gx & \text{Pull}\to \\
(fx \to gx) \to fx \to gx & \text{Gen}\exists, x' = x \\
1 & \text{Taut}
\end{array}
$$

j)

$(\forall x.\, fx \to gx) \to \exists f \to \exists g$

$(\forall x.\, fx \to gx) \to fx \to \exists g$ Pull $\to$, Gen$\forall$

$\exists x'.\, (\forall x.\, fx \to gx) \to fx \to gx'$ Pull $\to$

$(\forall x.\, fx \to gx) \to fx \to gx$ Gen$\exists, x' = x$

remaining steps as in 11.5 i)

k)

$(\forall x.\, fx \to \overline{gx}) \to \forall y.\, gy \to \overline{fy}$

$(\forall x.\, fx \to \overline{gx}) \to gy \to \overline{fy}$ Pull $\to$, Gen$\forall$

$\exists x.\, (fx \to \overline{gx}) \to gy \to \overline{fy}$ Pull $\to$

$(fy \to \overline{gy}) \to gy \to \overline{fy}$ Gen$\exists, x = y$

$1$ Taut

**Exercise 11.6 (Counterexample)** Let $\mathrm{QL} \vdash \forall f \vee \forall g = \forall x.\, fx \vee gx$. Then

$\mathrm{QL} \vdash (\forall x.x) \vee (\forall x.\bar{x}) = \forall x.\, (\lambda x.x)x \vee (\lambda x.\bar{x})x$ Generativity

$\mathrm{QL} \vdash (\forall x.x) \wedge 0 \vee (\forall x.\bar{x}) \wedge \bar{1} = \forall x.\, x \vee \bar{x}$ $\forall I, \mathrm{GR}, \beta$

$\mathrm{QL} \vdash 0 = \forall x.1$ Taut

$\mathrm{QL} \vdash 0 = 1$ $\forall 1$

Contradiction!

**Exercise 11.7 (Deductivity)**

$s = a \wedge f1$

$t = fa$

Property 2) is satisfied since

$$a \wedge f1 = 1 \overset{\mathrm{BA}}{\vdash} \{a = 1, f1 = 1\} \overset{\mathrm{BA}}{\vdash} fa = 1 \qquad \text{by And}$$

Property 3) is satisfied due to the following reason.

First, $\mathrm{BA} \vdash a \wedge f1 \to fa = \bar{a} \vee \overline{f1} \vee fa$.

Now consider a powerset algebra $\mathcal{A}$ over the set $\{1,2\}$ s.t.

$$\begin{aligned} \mathcal{A}a &= \{1\} \\ \mathcal{A}f\{1,2\} &= \{1,2\} \\ \mathcal{A}f\{1\} &= \emptyset \end{aligned}$$

BA $\not\vdash a \wedge f1 \rightarrow fa = 1$ now follows from the fact that $\mathcal{A}$ does not satisfy $\bar{a} \vee \overline{f1} \vee fa = 1$.