Introduction to Computational Logic, SS 2006: Solution for assignment 10

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Exercise 10.1 (Warm-up)

$\vdash$:

$\bar{s} = \bar{t}$	$s = t$

$\dashv$:

$s = \bar{\bar{s}}$	Double negation
$= \bar{\bar{t}}$	$\bar{s} = \bar{t}$
$= t$	Double negation

Exercise 10.2 (Think Semantically) No, the equation system has no solution. Since $\{x, y, z\} \subseteq \{0, 1\}$, at least two of the variables must be equal. Hence at least one equation is not satisfied. You can verify this by constructing the prime tree that represents the solution for the equation system.

Exercise 10.3 (BCA and Conversion Proofs)

a)

$fx \wedge fy \to f(x \wedge y)$

x

$f0 \wedge fy \to f0$		$f1 \wedge fy \to fy$	
1	Taut ($x \wedge y \to x = 1$)	1	Taut ($x \wedge y \to y = 1$)

b)

$$
\begin{aligned}
1 &= t0 \to t1 \to tx && \text{BCA}\\
&= (f0 \wedge fy \to f(0 \wedge y)) \to t1 \to tx && \beta\\
&= (f0 \wedge fy \to f0) \to t1 \to tx && \text{Taut } (0 \wedge y = 0)\\
&= 1 \to t1 \to tx && \text{Taut } (x \wedge y \to x = 1)\\
&= t1 \to tx && \text{I1}\\
&= (f1 \wedge fy \to f(1 \wedge y)) \to tx && \beta\\
&= (f1 \wedge fy \to fy) \to tx && \text{Taut } (1 \wedge y = y)\\
&= 1 \to tx && \text{Taut } (x \wedge y \to y = 1)\\
&= tx && \text{I1}\\
&= fx \wedge fy \to f(x \wedge y) && \beta
\end{aligned}
$$

Exercise 10.4 (Dualisation)

a) $f1 \not\leftarrow (f0 \not\leftarrow fx) = 0$

b) By Eq it suffices to show PL $\vdash f1 \not\leftarrow (f0 \not\leftarrow fx) \leftrightarrow 0 = 1$.

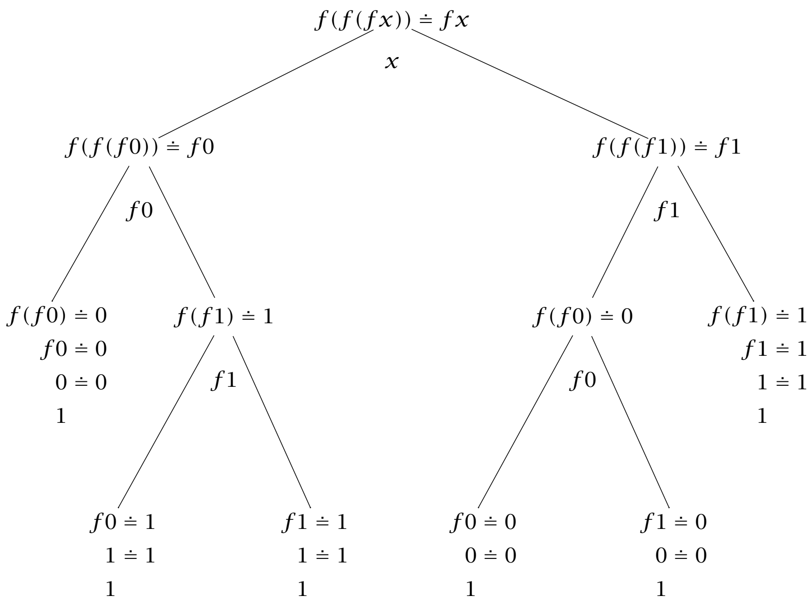
Since $x \not\leftarrow (y \not\leftarrow z) \leftrightarrow 0 = x \vee y \vee \bar{z}$ is a tautology†, it suffices to show PL $\vdash f1 \vee f0 \vee \overline{fx} = 1$.

By BCA it suffices to show PL $\vdash f1 \vee f0 \vee \overline{f0} = 1$, PL $\vdash f1 \vee f0 \vee \overline{f1} = 1$
Since both equations are in Taut, we are done.

†) Here is a proof sketch for the tautology:

$$
\begin{aligned}
x \not\leftarrow (y \not\leftarrow z) \leftrightarrow 0 &= (y \not\leftarrow z) \to x\\
&= (z \to y) \vee x\\
&= \bar{z} \vee y \vee x
\end{aligned}
$$

Exercise 10.5 (Kaminski's Equation)



Exercise 10.6 (Deductivity)

$$A = \emptyset$$
$$s = x \doteq y$$
$$t = 0$$

since

$$A, \{x \doteq y = 1\} \overset{\text{PL}}{\vdash} 0 = 1$$
$$A \overset{\text{PL}}{\nvdash} x \doteq y \to 0 = 1$$

Exercise 10.7 (Boolean Quantifiers)

a) $\vdash$:

$$\begin{aligned} \forall g &= g0 \wedge g1 \\ &= 1 \wedge 1 \\ &= 1 \end{aligned} \qquad \begin{aligned} \forall f &= f0 \wedge f1 \\ gx &= 1 \\ \text{Taut } (x \wedge x &= x) \end{aligned}$$

$\dashv$:

$$
\begin{aligned}
1 &= g0 \to g1 \to gx && \text{BCA} \\
&= g0 \wedge g1 \to gx && \text{Schönfinkel} \\
&= \forall g \to gx && \forall f = f0 \wedge f1 \\
&= 1 \to gx && \forall g = 1 \\
&= gx && \text{I1}
\end{aligned}
$$

b) Let $h = (\doteq)$.
Then for all y: $\exists x . x \doteq y = 1$, but not $\exists x . \forall y . x \doteq y = 1$.
Hence $\exists x \forall y . x \doteq y = 1 \overset{\text{PL}'}{\not\dashv} \exists x . x \doteq y = 1$ follows from soundness.