# Introduction to Computational Logic, SS 2006: Solution for assignment 9

Prof. Dr. Gert Smolka, Dipl.-Inform. Mathias Möhl

**Exercise 9.1** $A \vdash x \;:\Longleftrightarrow\; A \subseteq \mathbb{N} \wedge x \in \mathbb{N} \wedge (x \notin A \Rightarrow A \text{ infinite})$

**Exercise 9.2**

"$\Longrightarrow$"

| | | |
|---|---|---|
| Let | $A \vdash A'$ | (1) |
| | $x \in X$ | (2) |
| | $A' \vdash x$ | (3) |
| Then | $A, A' \vdash x$ | (3) and Monotonicity |
| | $A \vdash x$ | (1) and Idempotence |

"$\Longleftarrow$"

| | | |
|---|---|---|
| Let | $\forall x \in X\colon\; A' \vdash x \Rightarrow A \vdash x$ | (1) |
| | $y \in A'$ | (2) |
| Then | $A' \vdash y$ | (2) and Expansivity |
| | $A \vdash y$ | (1) |
| Hence | $A \vdash A'$ | |

**Exercise 9.3** Let $Q = \{x \in \mathbb{N} \mid \exists n \in \mathbb{N}\colon\; x = 3n\}$. We show $S[\{3, 15, 21\}] \subseteq Q$ with the Closure Theorem. We have to show:

1) $\{3, 15, 21\} \subseteq Q$

2) $x, y \in Q \Rightarrow x + y \in Q$

Proof obligation (1) is obvious. For (2) let $x, y \in Q$. Then $x = 3 \cdot n_1$ and $y = 3 \cdot n_2$. Hence $x + y = 3(n_1 + n_2) \in Q$.

**Exercise 9.4 (Incompleteness)**

$$A = \{f0 = 1, f(\sigma 0) = 1, f(\sigma(\sigma 0)) = 1, \dots\}$$
$$e = \{fx = 1\}$$

Then $\text{Nat}, A \vDash e$. We now show that $\text{Nat}, A \vdash e$ leads to a contradiction:

| | |
|---|---|
| Suppose | $\text{Nat}, A \vdash e$ |
| Then | $\text{Nat}, A' \vdash e$ for some finite $A' \subseteq A$ |
| | $\text{Nat}, A' \vDash e$ by soundness |

This is a contradiction since there is a model of Nat,A' that does not satisfy $e$.

**Exercise 9.5 (Termination)**

a) Termination of r can be expressed with the following equation:

$$\forall x.\neg\exists f.fx \wedge (\forall y.fy \to (\exists z.fz \wedge ryz)) = 1$$

Since PL does not provide the constants $\neg$ and $\exists$: $((V \to B) \to B) \to B$, we choose the following equivalent equation:

$$\text{ter} \;\; = \;\; fx \wedge \forall y.fy \to \exists z.fz \wedge ryz = 0$$

b) $A = \{ra_1a_2 = 1, ra_2a_3 = 1, \dots\}$

**Exercise 9.6 (Finiteness)**

a) $\text{injective} \; = \; \lambda f.(fx \doteq fy) \to (x \doteq y)$
$\text{surjective} \; = \; \lambda f.\forall y.\exists x.\; fx \doteq y$
$\text{fin} \; = \; (\text{injective } f \to \text{surjective } f = 1)$

b) $A = \{\, a_i \doteq a_j = 0 \mid 1 \leq i < j \,\}$

**Exercise 9.7 (Transitive Closure)**

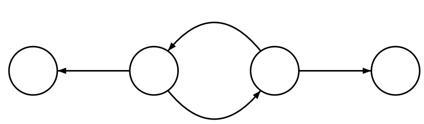
a) $\text{trans} = \lambda r.\forall xyz.\; rxy \wedge ryz \to rxz$

b) $\text{incl} = \lambda rr'.\forall xy.\; r'xy \to rxy$

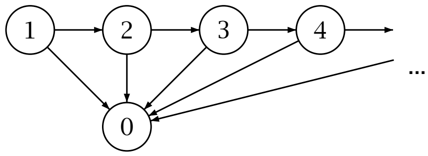
c) $\text{tc} = \lambda rr'.\text{trans } r' \wedge \text{incl } rr' \wedge \forall r''.(\text{trans } r'' \wedge \text{incl } rr'') \to \text{incl } r'r''$

d) Let $r$ be the identity relation. Then the equation $r'xy = rxy \vee (\exists z.rxz \wedge r'zy)$ is satisfied by every $r' \supseteq r$.

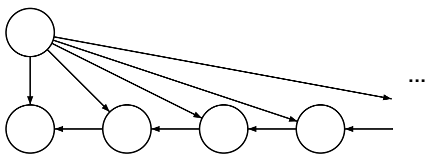
**Exercise 9.8**



**Exercise 9.9**




**Exercise 9.10**




**Exercise 9.12**

a) Let $y, z$ be normal forms of x. We show $y = z$.

| | |
|---|---|
| $x \to^* y$ and $x \to^* z$ | $y, z$ NF of $x$ |
| $y \to^* u$ and $z \to^* u$ for some $u$ | $\to$ confluent |
| $y = u$ and $z = u$ | $y, z$ NF |
| $y = z$ | |

b) Let $x \leftrightarrow^* y$. Then

| | |
|---|---|
| $x \to^* z$ and $y \to^* z$ for some z | $\to$ confluent, (i) |
| $z \to^* u$ for some NF u | $\to$ terminating, (ii) |
| $u$ NF of $x$ and $y$ | Transitivity of $\to^*$ |

The claim now follows from (a).