Introduction to Computational Logic, SS 2006: Solution for assignment 8

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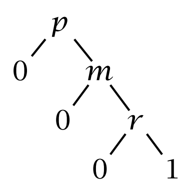
Exercise 8.1 (Modelling)

a) (i) $r \to p \wedge m = 1$

(ii) $\bar{r} \to \bar{p} \wedge c = 1$

(iii) $\bar{p} \to \bar{c} = 1$

b) Prime tree:



Exercise 8.2

$$\begin{aligned} 0 &= a\bar{a} && \text{Comp} \\ &= aa && a = \bar{a} \\ &= a && \text{Idem} \\ &= a + a && \text{Idem} \\ &= a + \bar{a} && a = \bar{a} \\ &= 1 && \text{Comp} \end{aligned}$$

Exercise 8.3

$$\begin{aligned} \lambda x.\, 0 \to x &= \lambda x.1 \\ \lambda x.\, 1 \to x &= \lambda x.x \\ \lambda f x.\, f0 \to f1 \to fx &= \lambda f x.1 \end{aligned}$$

Exercise 8.4

$$\begin{aligned} xy &= xy1 && \text{Id} \\ &= xy0 && 0 = 1 \\ &= 0 && \text{Dom} \end{aligned}$$

$$\begin{aligned} x + y &= x + y + 0 && \text{Id} \\ &= x + y + 1 && 0 = 1 \\ &= 1 && \text{Dom} \end{aligned}$$

$$\begin{aligned} 1 &= 1 \cdot 1 && \text{Id} \\ &= 0 && xy = 0 \end{aligned}$$

Exercise 8.5 (Closed Specifications) We give the proof twice, in schematic and prosaic form.

schematic proof:

Let A be closed.
"$\Leftarrow$" Let

$$\begin{aligned} A &\models e && (1) \\ \mathcal{I} &\models A && (2) \end{aligned}$$

To show: $\mathcal{I} \models e$
Let $\mathcal{A} \subseteq \mathcal{I}, \mathcal{N}A \subseteq Dom\,\mathcal{A}$. $\mathcal{A}$ exists since A is closed. Then

$$\begin{aligned} \mathcal{A} &\models A && (2),\ \text{coincidence} \\ \mathcal{A} &\models e && (1) \\ \mathcal{I} &\models e && \text{Def. } \mathcal{A} \models e \end{aligned}$$

"$\Longrightarrow$" Let

$$\forall \mathcal{I}:\ \mathcal{I} \vDash A \Longrightarrow \mathcal{I} \vDash e \qquad (1)$$
$$\mathcal{A} \vDash A \qquad (2)$$
$$\mathcal{A} \subseteq \mathcal{I} \qquad (3)$$

To show: $\mathcal{I} \vDash e$

$$\mathcal{I} \vDash A \qquad (2),(3)$$
$$\mathcal{I} \vDash e \qquad (1)$$

prosaic proof:

We show the equivalence for each direction separately.

"$\Longrightarrow$" : Let $A \vDash e$ and consider an arbitrary $\mathcal{I}$ that satisfies $\mathcal{I} \vDash A$. We need to show $\mathcal{I} \vDash e$.
Since A is closed, $\mathcal{N}A$ contains no variables. Hence, there exists some $\mathcal{A} \subseteq \mathcal{I}$ with *Dom* $\mathcal{A} = \mathcal{N}A$. By coincidence of $\mathcal{I}$ and $\mathcal{A}$, $\mathcal{I} \vDash A$ implies $\mathcal{A} \vDash A$. Furthermore, $\mathcal{A} \vDash A$ together with $A \vDash e$ implies $\mathcal{A} \vDash e$. Finally we can conclude $\mathcal{I} \vDash e$ by $\mathcal{A} \subseteq \mathcal{I}$ and the definition of $\mathcal{A} \vDash e$.

"$\Longleftarrow$": Let A, e be such that

$$(1) \quad \forall \mathcal{I}:\ \mathcal{I} \vDash A \Longrightarrow \mathcal{I} \vDash e$$

and consider some $\mathcal{A}, \mathcal{I}$ such that

$$(2) \quad \mathcal{A} \vDash A \qquad \text{and} \qquad (3) \quad \mathcal{A} \subseteq \mathcal{I} \quad (3)$$

We have to show that $\mathcal{I} \vDash e$ since this implies $\mathcal{A} \vDash e$ and finally $A \vDash e$.
(2) and (3) imply $\mathcal{I} \vDash A$ by definition of $\mathcal{A} \vDash A$. Together with (1) this implies $\mathcal{I} \vDash e$.

Exercise 8.6

$$\begin{aligned}
s &= t \\
(\lambda f x.f)s &= (\lambda f x.f)t && \text{CR} \\
(\lambda f x.f)s &= \lambda x.s && \beta \\
(\lambda f x.f)t &= \lambda x.t && \beta \\
\lambda x.s &= (\lambda f x.f)s && \text{Sym} \\
\lambda x.s &= (\lambda f x.f)t && \text{Trans} \\
\lambda x.s &= \lambda x.t && \text{Trans}
\end{aligned}$$

Exercise 8.7

a)

$$\begin{aligned}
s &= (\lambda x.s)y && \beta \\
&= (\lambda x.t)y && \lambda x.s = \lambda x.t \\
&= t && \beta
\end{aligned}$$

$$\lambda x.s = \lambda x.t \qquad\qquad s = t,\ \xi$$

b) Proof by induction on $n = |\mathcal{N}e \cap Var|$.
Case $n = 0$: The claim holds with $e = e'$.
Case $n > 0$: Let $x \in \mathcal{N}e \cap Var$, $e = (s,t)$ and $e'' = (\lambda x.s, \lambda x.t)$. Then $\{e\} \vdash\dashv \{e''\}$ and $|\mathcal{N}e \cap Var| > |\mathcal{N}e'' \cap Var|$. Hence there exists a closed e' such that $\{e''\} \vdash\dashv \{e'\}$ by induction hypothesis. This proves the claim since $\{e\} \vdash\dashv \{e'\}$ by transitivity of $\vdash\dashv$.

Exercise 8.8 (Semantic Entailment)

a) Let

$$\begin{aligned}
A &= \{x = y\} \\
e &= \{0 = 1\} \\
\mathcal{I}x &= \mathcal{I}y = \mathcal{I}0 \neq \mathcal{I}1
\end{aligned}$$

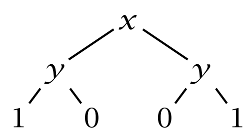
Then $A \vDash e$ and $\mathcal{I} \vDash A$, but $\mathcal{I} \nvDash e$.

b) A is closed (See Ex. 8.5)

Exercise 8.9 (Significant Variables) The prime tree of $(\bar{x}+y)(x+y)(y+z)$ is $(y, 0, 1)$. Hence y is the only significant variable.

Exercise 8.10 (Boolean Equation Systems)

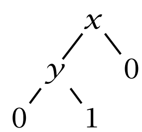
i) a)



b)

$$0$$

c)



d)

$$1$$

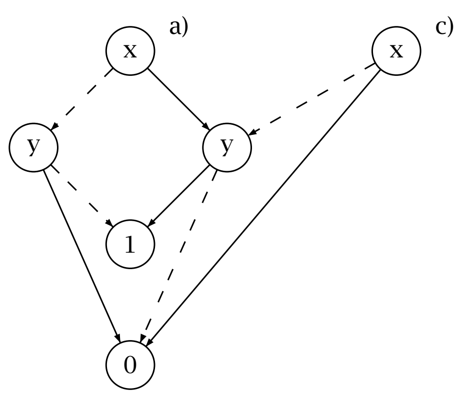
ii) a) $\sigma = \{(x, 0), (y, 1)\}$

b) Every σ has the required property.

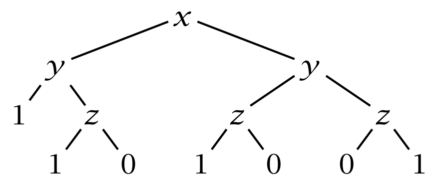
c) $\sigma = \{(x, 0), (y, 0)\}$

d) No such σ exists.

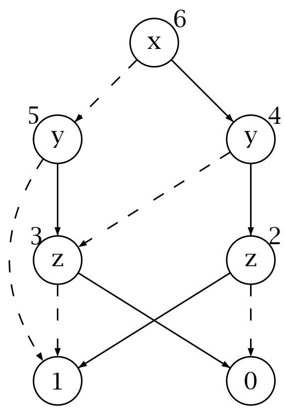
iii)



Exercise 8.11 (Decision Graph) a)



b)



c)

2	(z,0,1)
3	(z,1,0)
4	(y,3,2)
5	(y,1,3)
6	(x,5,4)