Introduction to Computational Logic, SS 2006: Solution for assignment 6

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Exercise 6.1 The powerset of a set with n elements has 2^n elements. Since $\forall n : 2^n \neq 7$ no powerset algebra has 7 elements. Stone's representation theorem hence implies that there exists no Boolean algebra with 7 elements.

Exercise 6.2 The powerset algebra of a set with 3 elements has 8 elements. We choose a set $\{a, b, c\}$. Then

$$\begin{aligned}
\mathcal{A}B &= \mathcal{P}\{a,b,c\} \\
\mathcal{A}0 &= \emptyset \\
\mathcal{A}1 &= \{a,b,c\} \\
\mathcal{A}{\cdot}vw &= v \cap w \\
\mathcal{A}{+}vw &= v \cup w \\
\mathcal{A}\bar{\ }v &= \{a,b,c\} - v
\end{aligned}$$

Exercise 6.3 $\theta = \{0{:=}1,\ 1{:=}0,\ +{:=}\cdot,\ \cdot{:=}+\}$

Exercise 6.4

Absorption	$x(x+y) = x$	$x + xy = x$
Complements	$x\bar{x} = 0$	$x + \bar{x} = 1$
Distributivity	$x(y+z) = xy + xz$	
	$x + yz = (x+y)(x+z)$	
Dominance	$0x = 0$	$1 + x = 1$
Idempotence	$xx = x$	$x + x = x$
Identities	$x1 = x$	$x + 0 = x$
Resolution	$xy + \bar{x}z = xy + \bar{x}z + yz$	
	$(x+y)(\bar{x}+z) = (x+y)(\bar{x}+z)(y+z)$	

Exercise 6.5

a)

$$\begin{aligned} xx &= xx + 0 && \text{Identities} \\ &= xx + x\bar{x} && \text{Complements} \\ &= x(x + \bar{x}) && \text{Distributivity} \\ &= x1 && \text{Complements} \\ &= x && \text{Identities} \end{aligned}$$

b)

$$\begin{aligned} 0x &= 0x + 0 && \text{Identities} \\ &= 0x + x\bar{x} && \text{Complements} \\ &= x(0 + \bar{x}) && \text{Distributivity} \\ &= x\bar{x} && \text{Identities} \\ &= 0 && \text{Complements} \end{aligned}$$

c)

$$\begin{aligned} x &= x + 0 && \text{Identities} \\ &= x + 0y && \text{(b)} \\ &= (x + 0)(x + y) && \text{Distributivity} \\ &= x(x + y) && \text{Identities} \end{aligned}$$

d)

$$\begin{aligned} xy + \bar{x}z &= x(x + z)y + \bar{x}(\bar{x} + y)z && \text{(c)} \\ &= xxy + xyz + \bar{x}\bar{x}z + \bar{x}yz && \text{Distributivity} \\ &= xy + xyz + \bar{x}z + \bar{x}yz && \text{(a)} \\ &= xy + \bar{x}z + (x + \bar{x})yz && \text{Distributivity} \\ &= xy + \bar{x}z + 1yz && \text{Complements} \\ &= xy + \bar{x}z + yz && \text{Identities} \end{aligned}$$

Exercise 6.6

$$\begin{aligned} x &= 1x && \text{Identities} \\ &= 0x && 0 = 1 \\ &= 0 && \text{Dominance} \\ &= 0y && \text{Dominance} \\ &= 1y && 0 = 1 \\ &= y && \text{Identities} \end{aligned}$$

Exercise 6.7

$\vdash$:

$$
\begin{aligned}
b &= 1b && \text{Identities} \\
&= (a+\bar{a})b && \text{Complements} \\
&= ab + \bar{a}b && \text{Distributivity} \\
&= 0 + \bar{a}b && \text{ab=0} \\
&= a\bar{a} + \bar{a}b && \text{Complements} \\
&= (a+b)\bar{a} && \text{Distributivity} \\
&= 1\bar{a} && \text{a+b=1} \\
&= \bar{a} && \text{Identities}
\end{aligned}
$$

$\dashv$:

$$
\begin{aligned}
a + b &= a + \bar{a} && b = \bar{a} \\
&= 1 && \text{Complements}
\end{aligned}
$$

$$
\begin{aligned}
ab &= a\bar{a} && b = \bar{a} \\
&= 0 && \text{Complements}
\end{aligned}
$$

Exercise 6.8 a) UoC yields for $\{b := x,\ a := \bar{x}\}$

$$\{\bar{x}x = 0,\ \bar{x} + x = 1\} \overset{\mathsf{BA}}{\vdash\dashv} \{x = \bar{\bar{x}}\}$$

Since the equations on the left are deducible from BA, we have

$$\emptyset \overset{\mathsf{BA}}{\vdash\dashv} \{x = \bar{\bar{x}}\}$$

and therefore $\mathsf{BA} \vdash x = \bar{\bar{x}}$

b) Use $\{a := 0,\ b := 1\}$. Then we have $\{01 = 0,\ 0+1 = 1\} \overset{\mathsf{BA}}{\vdash\dashv} \{1 = \bar{0}\}$. Again, the equations on the left are deducible from BA, and therefore $\mathsf{BA} \vdash \bar{0} = 1$.

c) Use $\{a := xy,\ b := \bar{x} + \bar{y}\}$. Then

$$\{xy(\bar{x} + \bar{y}) = 0,\ xy + (\bar{x} + \bar{y}) = 1\} \overset{\mathsf{BA}}{\vdash\dashv} \{\bar{x} + \bar{y} = \overline{xy}\}$$

Now we show the equations on the left:

$$\begin{aligned} xy(\bar{x}+\bar{y}) &= xy\bar{x}+xy\bar{y} && \text{Distributivity} \\ &= 0y+x0 && \text{Complements} \\ &= 0(x+y) && \text{Distributivity} \\ &= 0 && \text{(6.5b)} \end{aligned}$$

and

$$\begin{aligned} xy+(\bar{x}+\bar{y}) &= (x+\bar{x}+\bar{y})(y+\bar{x}+\bar{y}) && \text{Distributivity} \\ &= (1+\bar{y})(1+\bar{x}) && \text{Complements} \\ &= (1(1+\bar{y}))(1(1+\bar{x})) && \text{Identities} \\ &= 11 && \text{(6.5c)} \\ &= 1 && \text{Identities} \end{aligned}$$

d) Use $\{a := x+y,\ b := \bar{x}\bar{y}\}$. Then

$$\{(x+y)\bar{x}\bar{y}=0,\ x+y+\bar{x}\bar{y}=1\} \overset{\mathsf{BA}}{\vdash\dashv} \{\bar{x}\bar{y}=\overline{x+y}\}$$

Now we show the equations on the left:

$$\begin{aligned} (x+y)\bar{x}\bar{y} &= x\bar{x}\bar{y}+y\bar{x}\bar{y} && \text{Distributivity} \\ &= 0\bar{y}+0\bar{x} && \text{Complements} \\ &= 0(x+y) && \text{Distributivity} \\ &= 0 && \text{(6.5b)} \end{aligned}$$

and

$$\begin{aligned} x+y+\bar{x}\bar{y} &= (x+y+\bar{x})(x+y+\bar{y}) && \text{Distributivity} \\ &= (1+y)(1+x) && \text{Complements} \\ &= (1(1+y))(1(1+x)) && \text{Identities} \\ &= 11 && \text{(6.5c)} \\ &= 1 && \text{Identities} \end{aligned}$$

Note the similarity between these proofs and the proofs of their duals in part c) of this exercise.

Exercise 6.9

a) $\vdash$:

$$\begin{aligned} ab &= 1b && a = 1 \\ &= 11 && b = 1 \\ &= 1 && \text{Identities} \end{aligned}$$

$\dashv$:

$$\begin{aligned} a &= a(a+b) && \text{Absorption} \\ &= aa + ab && \text{Distributivity} \\ &= aa + 1 && ab = 1 \\ &= 1 && \text{Dominance} \end{aligned}$$

b analogously

b) $\vdash$:

$$\begin{aligned} a + b &= 0 + b && a = 0 \\ &= 0 + 0 && b = 0 \\ &= 0 && \text{Identities} \end{aligned}$$

$\dashv$:

$$\begin{aligned} a &= a(a+b) && \text{Absorption} \\ &= a0 && x + y = 0 \\ &= 0 && \text{Dominance} \end{aligned}$$

b analogously

c) $\vdash$:

$$\begin{aligned} b &= b + ba && \text{Absorption} \\ &= b + a && a = ab \end{aligned}$$

$\dashv$:

$$\begin{aligned} a &= a(a+b) && \text{Absorption} \\ &= ab && b = b + a \end{aligned}$$

d) $\vdash$:

$$\begin{aligned} a &= 1a && \text{Identities} \\ &= (\bar{a} + b)a && \bar{a} + b = 1 \\ &= a\bar{a} + ab && \text{Distributivity} \\ &= 0 + ab && \text{Complements} \\ &= ab && \text{Identities} \end{aligned}$$

$\dashv$:

$$\begin{aligned} \bar{a} + b &= \overline{ab} + b && a = ab \\ &= \bar{a} + \bar{b} + b && \text{de Morgan} \\ &= \bar{a} + 1 && \text{Complements} \\ &= 1 && \text{Dominance} \end{aligned}$$

e) $\dashv$:

$$\begin{aligned} (\bar{a} + b)(a + \bar{b}) &= (\bar{a} + a)(a + \bar{a}) && a = b \\ &= 11 && \text{Complements} \\ &= 1 && \text{Identities} \end{aligned}$$

$\vdash$:

$$\begin{aligned} & (\bar{a} + b)(a + \bar{b}) = 1 \\ \vdash\ & \bar{a} + b = 1,\ a + \bar{b} = 1 && \text{(a)} \\ \vdash\ & a = ab,\ b = ba && \text{(d)} \\ \vdash\ & a = b \end{aligned}$$

Exercise 6.10

$$\begin{aligned} a = b,\ c = d &\overset{\text{BA}}{\vdash\dashv} (\bar{a} + b)(a + \bar{b}) = 1,\ (\bar{c} + d)(c + \bar{d}) = 1 && \text{Exercise 6.9 e)} \\ &\overset{\text{BA}}{\vdash\dashv} (\bar{a} + b)(a + \bar{b})(\bar{c} + d)(c + \bar{d}) = 1 && \text{Exercise 6.9 a)} \end{aligned}$$

Hence $s = (\bar{a} + b)(a + \bar{b})(\bar{c} + d)(c + \bar{d})$.

Exercise 6.11

a)

$$
\begin{aligned}
(\lambda f.fx\bar{x})(+) &= x + \bar{x} && \beta \\
&= 1 && \text{Complements}
\end{aligned}
$$

b)

$$
\begin{aligned}
(\lambda xy.\bar{x} + y)0 &= \lambda y.\bar{0} + y && \beta \\
&= \lambda y.1 + y && \text{Negated Identities} \\
&= \lambda y.1 && \text{Dominance} \\
&= \lambda x.x + \bar{x} && \text{Complements}
\end{aligned}
$$

c)

$$
\begin{aligned}
&(\lambda f.f(fxy)y)(\lambda xy.\bar{x} + y) \\
&= (\lambda xy.\bar{x} + y)(\bar{x} + y)y && \beta \\
&= \overline{\bar{x} + y} + y && \beta \\
&= \bar{\bar{x}}\bar{y} + y && \text{de Morgan} \\
&= x\bar{y} + y && \text{Double negation} \\
&= (x + y)(\bar{y} + y) && \text{Distributivity} \\
&= (x + y)1 && \text{Complements} \\
&= x + y && \text{Identities}
\end{aligned}
$$