Introduction to Computational Logic, SS 2006: Solution for assignment 5

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Exercise 5.1 (Renaming)

a) $y \notin \mathcal{N}t \implies (t[x{:=}y])[y{:=}s] = t[x{:=}s]$

Proof by induction on $|t|$. Case analysis according to Par.

Case $t = x$. Then

$$\begin{aligned}(x[x := y])[y := s] &= y[y := s] && \text{SN}\\ &= s && \text{SN}\\ &= x[x := s] && \text{SN}\end{aligned}$$

Case $t = u$, $u \neq x$. Then

$$\begin{aligned}(u[x := y])[y := s] &= u[y := s] && \text{SN}, u \neq x\\ &= u && \text{SN}, y \notin \mathcal{N}(t)\\ &= u[x := s] && \text{SN}, u \neq x\end{aligned}$$

Case $t = s't'$. Then

$$\begin{aligned}&((s't')[x := y])[y := s]\\ &= ((s'[x := y])(t'[x := y]))[y := s] && \text{SA}\\ &= ((s'[x := y])[y := s])((t'[x := y])[y := s]) && \text{SA}\\ &= (s'[x := s])(t'[x := s]) && \text{induction hypothesis}\\ &= (s't')[x := s] && \text{SA}\end{aligned}$$

Case $t = \lambda z.s'$. By Proposition 2.2, Inf and IL we can assume $z \notin \mathcal{N}(s) \cup \{x, y\}$.

Hence

$$
\begin{aligned}
&((\lambda z.s')[x := y])[y := s] && \\
&= (\lambda z.(s'[x := y]))[y := s] && \text{SL, } z \notin \{x, y\} \\
&= \lambda z.((s'[x := y])[y := s]) && \text{SL, } z \neq y, z \notin \mathcal{N}(s) \\
&= \lambda z.(s'[x := s]) && \text{induction hypothesis} \\
&= (\lambda z.s')[x := s] && \text{SL, } z \neq x,\ z \notin \mathcal{N}(s)
\end{aligned}
$$

∎

b) $\forall t \forall v \forall \theta\colon\ v \in \mathcal{N}(\mathbf{S}\theta t) \iff (\exists u \in \mathcal{N}t\colon\ v \in \mathcal{N}(\mathbf{S}\theta u))$

Proof by induction on $|t|$. Case analysis according to Par.

Case $t = u$ Then

$$
v \in \mathcal{N}(\mathbf{S}\theta u) \iff \exists u \in \mathcal{N}u\colon\ v \in \mathcal{N}(\mathbf{S}\theta u)
$$

Case $t = ss'$. Then

$$
\begin{aligned}
&v \in \mathcal{N}(\mathbf{S}\theta(ss')) && \\
&\iff v \in \mathcal{N}((\mathbf{S}\theta s)(\mathbf{S}\theta s')) && \text{SA} \\
&\iff v \in \mathcal{N}(\mathbf{S}\theta s) \vee v \in \mathcal{N}(\mathbf{S}\theta s') && \text{NA} \\
&\iff \exists u \in \mathcal{N}s\colon\ v \in \mathcal{N}(\mathbf{S}\theta u) \vee \exists u \in \mathcal{N}s'\colon\ v \in \mathcal{N}(\mathbf{S}\theta u) && \text{ind. hyp.} \\
&iff \exists u \in \mathcal{N}(ss')\colon\ v \in \mathcal{N}(\mathbf{S}\theta u) && \text{NA}
\end{aligned}
$$

Case $t = \lambda x.s$. By Proposition 2.2, Inf and IL we can assume that $x \neq v$ and $\forall u \in \mathcal{N}t\colon\ x \notin \mathcal{N}(\theta u)$. Let $\theta' = \theta[x := x]$. Then

$$
\begin{aligned}
&v \in \mathcal{N}(\mathbf{S}\theta t) && \\
&\iff v \in \mathcal{N}(\lambda x.\mathbf{S}\theta' s) && \text{SL} \\
&\iff v \in \mathcal{N}(\mathbf{S}\theta' s) && \text{NL, } v \neq x \\
&\iff \exists u \in \mathcal{N}s\colon\ v \in \mathcal{N}(\theta' u) && \text{induction hypothesis} \\
&\iff \exists u \in \mathcal{N}s\colon\ u \neq x \wedge v \in \mathcal{N}(\theta' u) && v \neq x \\
&\iff \exists u \in \mathcal{N}s\colon\ u \neq x \wedge v \in \mathcal{N}(\theta u) && \\
&\iff \exists u \in \mathcal{N}t\colon\ v \in \mathcal{N}(\theta u) && \text{NL}
\end{aligned}
$$

∎

Exercise 5.2 (Boolean Connectives)

$$\begin{aligned}
\neg x &= x \to 0\\
x \wedge y &= \neg(x \to \neg y)\\
x \vee y &= \neg x \to y
\end{aligned}$$

or alternatively

$$\begin{aligned}
\neg 0 &= 1 & \neg 1 &= 0\\
0 \wedge x &= 0 & 1 \vee x &= 1\\
1 \wedge x &= x & 0 \vee x &= x
\end{aligned}$$

Exercise 5.3 (Quantifiers)

$$\begin{aligned}
\forall(\lambda x.1) &= 1 & \exists(\lambda x.0) &= 0\\
\forall f \to f x &= 1 & f x \to \exists f &= 1
\end{aligned}$$

Exercise 5.4 (Identity)

$$\begin{aligned}
x \doteq x &= 1\\
x \doteq y \to f x \to f y &= 1
\end{aligned}$$

Exercise 5.5 (Linear) Property 1) requires that every element of V must be reachable from o by σ. In other words, for all $v \in V$ there exists an n such that $v = \sigma^n o$. This is ensured by the induction axiom

$$f o \to (\forall x.\ f x \to f(\sigma x)) \to \forall f = 1$$

since for every unreachable v there is an f that holds on all reachable values but does not hold on v.

Exercise 5.6 (Infinite Chain)

$$o \doteq o = 1 \tag{1}$$
$$o \doteq \sigma x = 0 \tag{2}$$
$$\sigma x \doteq \sigma y = x \doteq y \tag{3}$$
$$x \doteq y = y \doteq x \tag{4}$$

The finite models of Linear do not satisfy these axioms due to the following reason. In the finite models σ is cyclic, that is, there exists an a such that $a = \sigma^n o = \sigma^{(n+m)} o$ for some n and $m \geq 1$. We show that this leads to the contradiction $1 = 0$:

$a \doteq a = (\sigma^n o \doteq \sigma^n o)$	$a \doteq a = (\sigma^n o \doteq \sigma^{(n+m)} o)$	
$= (o \doteq o)$	$= (o \doteq \sigma^m o)$	n times axiom (3)
$= 1$	$= 0$	axioms (1) and (2)

Alternatively, $\doteq$ can be axiomatized as in Exercise 5.4 and the following additional axioms ensure that σ is acyclic:

$(\sigma x \doteq \sigma y) = (x \doteq y)$	Each element has at most one predecessor
$(\sigma x \doteq o) = 0$	origin has no predecessor

Exercise 5.7 (Natural Numbers)

$$\forall(\lambda x.1) = 1$$
$$\forall f \to f x = 1$$

$$f \underline{0} \to (\forall(\lambda x. f x \to f(x+1))) \to \forall f = 1$$

$$\underline{0} \leq x = 1$$
$$x + \underline{1} \leq \underline{0} = 0$$
$$x + \underline{1} \leq y + \underline{1} = x \leq y$$

$$\underline{0} + y = y$$
$$(x + \underline{1}) + y = (x + y) + \underline{1}$$

$$\underline{0} \cdot y = \underline{0}$$
$$(x + \underline{1}) + y = x \cdot y + y$$

Note that $\lambda x.x+1$ serves as σ and the axioms for $\leq$ exclude finite models similar to the axioms for $\doteq$ in Exercise 5.6.