Introduction to Computational Logic, SS 2006: Solution for assignment 4

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Exercise 4.1 (Implicit Conditions)

IA $\forall s, s', t, t' \in \mathit{Ter}\colon\ s \in \mathit{Dom}\ A \wedge t \in \mathit{Dom}\ (As) \wedge s' \in \mathit{Dom}\ A \wedge t' \in \mathit{Dom}\ (As') \implies (Ast = As't' \iff s = s' \wedge t = t')$

IL $\forall t, t' \in \mathit{Ter}\quad \forall x, x' \in \mathit{Var}\colon\ \tau x = \tau x' \implies (Lxt = Lx't' \iff x' \notin \mathcal{N}(Lxt) \wedge t' = \mathbf{S}(\lambda y \in \mathit{Var}.\text{if } x = y \text{ then } x' \text{ else } y)t)$

SA $\forall f \in \mathit{Nam} \to \mathit{Ter}\quad \forall s, t \in \mathit{Ter}\colon\ t \in \mathit{Dom}\, A \wedge s \in \mathit{Dom}\,(At) \wedge f \in \mathit{Dom}\, \mathbf{S} \implies \mathbf{S}f(Ats) = A(\mathbf{S}ft)(\mathbf{S}fs)$

Exercise 4.2 (Construction)

$$\begin{aligned} \mathcal{N}u &= \{u\} \\ \mathcal{N}(st) &= \mathcal{N}s \cup \mathcal{N}t \\ \mathcal{N}(\lambda T.t) &= \mathcal{N}t \\ \mathcal{N}i &= \emptyset \end{aligned}$$

$$\begin{aligned} \mathbf{S}fu &= fu \\ \mathbf{S}f(st) &= (\mathbf{S}fs)(\mathbf{S}ft) \\ \mathbf{S}f(\lambda T.t) &= \lambda T.(\mathbf{S}ft) \\ \mathbf{S}fi &= i \end{aligned}$$

Exercise 4.3 (Trivial Substitution)

$$\forall t \in \mathit{Ter}\colon\ \mathbf{S}(\lambda u \in \mathit{Nam}.u)t = t$$

Proof By induction on $|t|$. Case analysis according to Par.

Case $t = u$. Then $\mathbf{S}(\lambda u \in \mathit{Nam}.u)u = u$ by SN.

Case $t = ss'$. Then

$$\begin{aligned} \mathbf{S}(\lambda u \in \mathit{Nam}.u)ss' &= (\mathbf{S}(\lambda u \in \mathit{Nam}.u)s)(\mathbf{S}(\lambda u \in \mathit{Nam}.u)s') && \text{SA} \\ &= ss' && \text{induction hypothesis} \end{aligned}$$

Case $t = \lambda x.s$. Then by NL $x \notin \mathcal{N}(\lambda x.s)$ and consequently $\forall u \in \mathcal{N}(\lambda x.s) : x \notin \mathcal{N}((\lambda u \in \mathit{Nam}.u)u) = \{u\}$. Hence

$$\begin{aligned} \mathbf{S}(\lambda u \in \mathit{Nam}.u)\lambda x.s &= \lambda x.(\mathbf{S}(\lambda u \in \mathit{Nam}.u)s) && \text{SL} \\ &= \lambda x.s && \text{induction hypothesis} \end{aligned}$$

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Exercise 4.4 (Coincidence)

$$\forall t \in \mathit{Ter}\ \ \forall f, g \in \mathit{Dom}\,\mathbf{S}\colon\ (\forall u \in \mathcal{N}t\colon\ fu = gu) \Longrightarrow \mathbf{S}ft = \mathbf{S}gt$$

Proof Proof by induction on $|t|$. Let $f, g \in \mathit{Dom}\,\mathbf{S}$ and let $\forall u \in \mathcal{N}t : fu = gu$ $(*)$. Case analysis according to Par.

Case $t = u$. Then $fu = gu$ by NN and $\mathbf{S}fu = \mathbf{S}fg$ by SN.

Case $t = ss'$. Then

$$\begin{aligned} \mathbf{S}f(ss') &= (\mathbf{S}fs)(\mathbf{S}fs') && \text{SA} \\ &= (\mathbf{S}gs)(\mathbf{S}gs') && \text{induction hypothesis} \\ &= \mathbf{S}g(ss') && \text{SA} \end{aligned}$$

Case $t = \lambda x.s$. By Proposition 2.1 of the skript ($\mathcal{N}t$ is a finite set), Inf and IL we can assume $\forall u \in \mathcal{N}(\lambda x.s) : x \notin \mathcal{N}(fu) \cup \mathcal{N}(gu)$. Hence

$$\begin{aligned} \mathbf{S}f(\lambda x.s) &= \lambda x.(\mathbf{S}(f[x := x])s) && \text{SL} \\ &= \lambda x.(\mathbf{S}(g[x := x])s) && \text{induction hypothesis, NL and } (*) \\ &= \mathbf{S}g(\lambda x.s) && \text{SL} \end{aligned}$$

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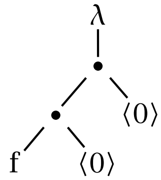
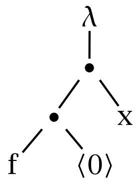
Exercise 4.5 (Counterexamples)

a) $t = fxx$, $t' = fx'x$

$\lambda x.t =$



$\lambda x'.t' =$



b) $t = fxx'$, $t' = fx'x'$