



**Introduction to Computational Logic, SS 2006:
Solution for assignment 1**

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Exercise 1.1 (Boolean Connectives)

a)

$$\begin{aligned}
 1 &= 0 \Rightarrow 0 \\
 \neg &= \lambda x \in \mathbb{B}. x \Rightarrow 0 \\
 \wedge &= \lambda x \in \mathbb{B}. \lambda y \in \mathbb{B}. \neg(x \Rightarrow \neg y) \\
 &= \lambda x \in \mathbb{B}. \lambda y \in \mathbb{B}. (x \Rightarrow (y \Rightarrow 0)) \Rightarrow 0 && \text{by expansion} \\
 \vee &= \lambda x \in \mathbb{B}. \lambda y \in \mathbb{B}. (\neg x) \Rightarrow y \\
 &= \lambda x \in \mathbb{B}. \lambda y \in \mathbb{B}. (x \Rightarrow 0) \Rightarrow y && \text{by expansion} \\
 \Leftrightarrow &= \lambda x \in \mathbb{B}. \lambda y \in \mathbb{B}. (x \Rightarrow y) \wedge (y \Rightarrow x) \\
 &= \lambda x \in \mathbb{B}. \lambda y \in \mathbb{B}. ((x \Rightarrow y) \Rightarrow ((y \Rightarrow x) \Rightarrow 0)) \Rightarrow 0 && \text{by expansion}
 \end{aligned}$$

b)

$$\begin{aligned}
 0 &= \neg 1 \\
 \wedge &= \lambda x \in \mathbb{B}. \lambda y \in \mathbb{B}. \neg(\neg x \vee \neg y) \\
 \Rightarrow &= \lambda x \in \mathbb{B}. \lambda y \in \mathbb{B}. \neg x \vee y
 \end{aligned}$$

Exercise 1.2 (Sets as functions)

- a) $\cap = \lambda f \in P(X). \lambda g \in P(X). \lambda x \in X. fx \wedge gx$
- b) $\cup = \lambda f \in P(X). \lambda g \in P(X). \lambda x \in X. fx \vee gx$
- c) $- = \lambda f \in P(X). \lambda g \in P(X). \lambda x \in X. fx \wedge \neg(gx)$
- d) $\subseteq = \lambda f \in P(X). \lambda g \in P(X). \forall x \in X: \neg(fx) \vee gx$
- e) $\parallel = \lambda f \in P(X). \lambda g \in P(X). \forall x \in X: \neg(fx \wedge gx)$
- f) $\in = \lambda x \in X. \lambda S \in P(X). Sx$

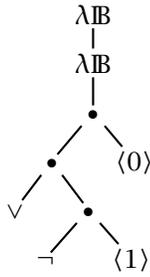
Exercise 1.3 (Identities and Quantifiers)

- a) $\forall_X = \lambda f \in X \rightarrow \mathbb{B}. f \doteq_{X \rightarrow \mathbb{B}} (\lambda x \in X. 1)$
(In the exercise we forgot to mention that you could use the constant 1.)
- b) $\exists_X = \lambda f \in X \rightarrow \mathbb{B}. \neg \forall x \in X. \neg(fx)$
- c) $\forall_X = \lambda f \in X \rightarrow \mathbb{B}. \neg \exists x \in X. \neg(fx)$
- d) $\doteq_{X \rightarrow Y} = \lambda f \in X \rightarrow Y. \lambda g \in X \rightarrow Y. \forall x \in X. fx \doteq_Y gx$
- e) $\doteq_{\mathbb{B}} = \Leftrightarrow.$

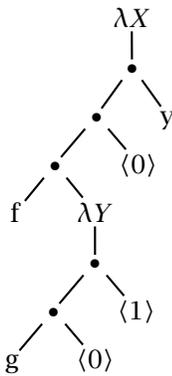
$$f) \dot{=} x = \lambda x \in X. \lambda y \in X. \forall f \in X \rightarrow \mathbb{B}. fx \Rightarrow fy$$

Exercise 1.4 (Tree Representations)

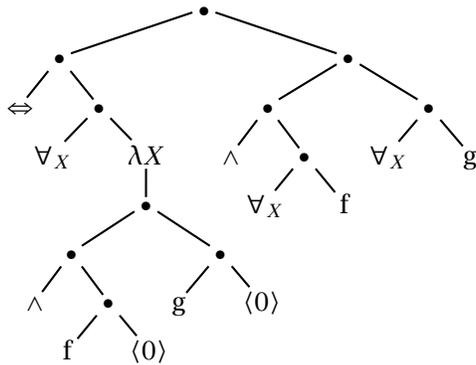
a) $\lambda x : \mathbb{B}. \lambda y : \mathbb{B}. \neg x \vee y$:



b) $\lambda x : X. f(\lambda y : Y. gyx)xy$:



c) $(\forall x : X. fx \wedge gx) \Leftrightarrow \forall f \wedge \forall g$



Exercise 1.5 (Substitution)

- a) $(fx(\lambda x : X. y))[x := y] = fy(\lambda x : X. y)$
- b) $(\lambda x : X. fxy)[y := x] = \lambda x' : X. fx'x$
- c) $(\lambda x : X. \lambda y : X. fxy)[f := gxy] = \lambda x' : X. \lambda y' : X. (gxy)x'y'$

Exercise 1.6 (Term Equality)

- a) λX
|
 $\langle 0 \rangle$
- b) λX
|
 $\langle 0 \rangle$
- c) λX
|
x
- d) λX
|
 $\langle 0 \rangle$
- e) λX
|
x
- f) λX
|
 λX
|
•
/ \ $\langle 0 \rangle$
f \ $\langle 1 \rangle$
- g) λX
|
 λX
|
•
/ \ $\langle 0 \rangle$
f \ $\langle 1 \rangle$

The terms in a), b) and d) are equal, the terms in c) and e) are equal and finally the terms in f) and g) are equal.