Midterm Exam Introduction to Computational Logic
SS 2006

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Name   Seat

Matriculation   Code

Please put your ID (or passport) and your student card on your desk.

Open the exam booklet only after you have been asked to do so. After you have opened the exam booklet, it is your obligation to check whether it is complete.

You may only use the exam booklet that carries your name and matriculation. You have to write the exam on the seat with the number that is printed on your exam booklet.

No auxiliary means are allowed. At your desk, you may only have writing utensils, beverages, food, and ID cards. Bags and jackets have to be left at the walls of the lecture room.

If you leave the room without turning in your exam booklet, then this will be judged as an attempt of deception.

If you need to go to the bathroom during the exam, please turn in your exam booklet. Only one person may go to the bathroom at a time.

All solutions have to be written on the right hand side pages of the exam booklet. The empty left hand side pages may serve as draft paper and **will not be graded**. No other paper is admitted. You may use a pencil.

The exam lasts 150 minutes. You can obtain at most 150 points. The number of points you can get for a problem gives you a hint about how much time you should spend on that problem. For passing the exam it is sufficient to obtain 75 points.

Every attempt of deception will force us to exclude you from this exam and all following exams of this course. The university keeps a record of attempts of deception.

1	2	3	4	5	6	7	8	9	10	Sum	**Grade**
12	11	24	12	20	10	11	16	14	20	150	

Problem 1. Tree representation (12 points)

Draw the tree representation of the term representation of the following formula. Represent bound variables with de Bruijn indices.

$$\forall x \in \mathbb{Z}\ \ \exists y \in \mathbb{Z}\colon\ x + y \leq a$$

Problem 2. Sets (2+9=11 points)

Let the following logical operations be given:

$$\begin{aligned} \neg &: B \to B \\ \wedge, \vee &: B \to B \to B \\ \forall &: (V \to B) \to B \end{aligned}$$

a) Give a type S whose members represent sets whose elements have type V.

$S =$

b) Give *closed* terms that describe the following set operations. Only use the variables $x : V$ and $X, Y : S$.

(i) intersection $(X \cap Y)$

(ii) membership $(x \in X)$

(iii) subset $(X \subseteq Y)$

Problem 3. Choice (3+3+6+4+8=24 points)

Let the following operations on Boolean values (B) and integers (Z) be given:

$$\begin{aligned} \neg :&\ B \to B \\ \wedge, \vee, \to :&\ B \to B \to B \\ + :&\ Z \to Z \to Z \\ \leq, \doteq :&\ Z \to Z \to B \end{aligned}$$

Moreover, let a choice function

$$C : (Z \to B) \to Z$$

be given that satisfies the equation

$$fx \to f(Cf) = 1$$

Find *closed* terms that describe the following objects. Only use the variables $x, y, z : Z$ and $f : Z \to B$.

a) $0 \in Z$.

b) Subtraction $Z \to Z \to Z$.

c) Minimum $Z \to Z \to Z$

d) Existential quantification $(Z \to B) \to B$.

e) Universal quantification $(Z \to B) \to B$.

Problem 4. Reduction and Elimination (12 points)

Let Elim be the specification

Constants

$$I\colon\ V \to V$$
$$K\colon\ V \to V \to V$$
$$S\colon\ (V \to V \to V) \to (V \to V) \to V$$

Axioms

$$Ix = x \qquad (I)$$
$$Kxy = x \qquad (K)$$
$$Sfg = \lambda x.fx(gx) \qquad (S)$$

Prove Elim $\vdash t = s$ for $t = (\lambda f x.fxx)(\lambda x.fx)$ and some combinatory term s.

Problem 5. Axiomatization (20 points)

Give a categorical specification for $\mathbb{B}$ and $\mathbb{N}$ with the following signature.

$0, 1:$	B	
$\to:$	$B \to B \to B$	
$\forall:$	$(N \to B) \to B$	
$\underline{0}:$	N	zero
$\sigma:$	$N \to N$	successor
$\doteq:$	$N \to N \to B$	identity

Problem 6. Evaluation (3+7=10 points)

Let $\mathcal{I}$ be an interpretation.

a) Let $x \in Var$ and $v \in \mathcal{I}(\tau x)$. Complete the following equations such that they define the interpretation $\mathcal{I}_{x,v}$.

$$\mathcal{I}_{x,v} T =$$

$$\mathcal{I}_{x,v} u =$$

b) Complete the following equations characterizing the evaluation function for $\mathcal{I}$.

$$\hat{\mathcal{I}} u =$$

$$\hat{\mathcal{I}}(st) =$$

$$\hat{\mathcal{I}}(\lambda x.s) =$$

Problem 7. Semantic Entailment (3+1+3+1+3=11 points)

State the following definitions.

a) $\mathcal{A}$ is a structure $:\Longleftrightarrow$

b) $\mathcal{I} \vDash s = t$ $:\Longleftrightarrow$

c) $\mathcal{A} \vDash e$ $:\Longleftrightarrow$

d) $\mathcal{A} \vDash E$ $:\Longleftrightarrow$

e) $A \vDash e$ $:\Longleftrightarrow$

Problem 8. Deduction Rules (6+10=16 points)

a) State the deduction rules ξ, β, and η.

b) Give a derivation of $st = s't$ from $s = s'$ that only employs the deduction rules Sym, Trans, CR and β.

Problem 9. Dualisation (14 points)

Prove $\mathsf{BA} \vdash \widehat{(x, s, t)} = (x, \hat{t}, \hat{s})$.

Use the following annotations: Def Cond, Def ˆ, Id, Compl, Dist, Res.

Problem 10. Prime Tree Algorithm (20 points)

Complete the following equations such that they describe a proceduce nand : $PT \to PT \to PT$ that returns $\pi(\overline{s \wedge t})$ for s, t.

cond $x\ s\ t$ =

nand $0\ s$ =

nand $s\ 0$ =

nand $1\ s$ =

nand $s\ 1$ =

nand $\underbrace{(x, s_0, s_1)}_{s=}\ \underbrace{(y, t_0, t_1)}_{t=}$ =

if $x = y$ then

else if $x < y$ then

else