Assignment 10
Introduction to Computational Logic, SS 2006

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Exercise 10.1 (Warm-up) Let s, t be terms of type B. Prove $s = t \overset{\text{BA}}{\vdash\dashv} \bar{s} = \bar{t}$ with two conversion proofs.

Exercise 10.2 (Think Semantically) Let x,y,z be 3 distinct variables. Is the Boolean equation system $\{x \leftrightarrow y = 0, x \leftrightarrow z = 0, y \leftrightarrow z = 0\}$ solvable?

Exercise 10.3 (BCA and Conversion Proofs)
Consider the term $s = fx \wedge fy \to f(x \wedge y)$ where $x, y\colon\ B$ and $f\colon\ B \to B$ are variables.

a) Prove $\text{PL} \vdash s = 1$ with BCA. Use a tree representation as in the proof of BRep on the slides from the lecture.

b) Find a conversion proof of $s = 1$ from $\text{PL} \cup \text{Taut}$ where

$$\text{Taut} \ := \ \{\, e \text{ pure} \mid \mathcal{T} \vDash e \,\}$$

Hint: Start from 1 and use $t = \lambda x.s$ for notation. For each conversion step with respect to Taut, annotate the equation from Taut that is employed.

Exercise 10.4 (Dualisation)

a) Give a dual of the axiom BCA.

b) Prove that the dual of BCA is deducible from PL. Use Eq, BCA and conversion proofs with respect to Taut.

Exercise 10.5 (Kaminski's Equation) Let $x\colon B$ and $f\colon B \to B$ be variables. Prove $\text{PL} \vdash f(f(fx)) = fx$ by Generalized BCA. Use the tree representation shown on the slides from the lecture.

Exercise 10.6 (Deductivity) Find A, s, t such that

1) $A, s = 1 \overset{\text{PL}}{\vdash} t = 1$

2) $A \overset{\text{PL}}{\nvdash} s \to t = 1$

Hint: Think about the implicit quantification of variables.

Exercise 10.7 (Boolean Quantifiers) Let PL' be PL extended with

$$\forall f = f0 \wedge f1$$
$$\exists f = f0 \vee f1$$

Let $g\colon B \to B$ and $h\colon B \to B \to B$ be constants. Prove the following:

a) $gx = 1 \overset{\text{PL}'}{\models\!\dashv} \forall g = 1$

b) $\exists x \forall y.hxy = 1 \overset{\text{PL}'}{\not\models\!\dashv} \exists x.hxy = 1$

Exercise 10.8 (Tautologies) Make sure that you are familiar with the following tautologies:

$1 \to x = x$	$x \to 1 = 1$	$x \to x = 1$
$0 \to x = 1$	$x \to 0 = \bar{x}$	$\overline{x \doteq y} = \bar{x} \doteq y$

$x \wedge y \to z = x \to y \to z$ (Schönfinkel)

$x \to y = \bar{y} \to \bar{x}$ (Contraposition)

$x \doteq 1 = x$	$x \doteq x = 1$	$x \doteq y = y \doteq x$
$x \doteq 0 = \bar{x}$	$x \doteq \bar{x} = 0$	$(x \doteq y) \doteq z = x \doteq (y \doteq z)$

$x \to y = x \leftrightarrow x \wedge y = y \leftrightarrow x \vee y$ (Golden Rule)

$x \doteq \bar{y} = \overline{x \wedge y} \wedge (x \vee y)$ (Uniqueness of Complements)