Assignment 9
Introduction to Computational Logic, SS 2006

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Exercise 9.1 Find an entailment relation on $\mathbb{N}$ that is not compact.

Exercise 9.2 Let $\vdash$ be an entailment relation on X. Prove:

$$A \vdash A' \iff \forall x \in X\colon\ A' \vdash x \Longrightarrow A \vdash x$$

Exercise 9.3 Consider the inference system

$$\{\, (\{x, y\}, x + y) \mid x, y \in \mathbb{N} \,\}$$

Prove that for every $x \in S[\{3, 15, 21\}]$ there exists an $n \in \mathbb{N}$ such that $x = 3n$.

Exercise 9.4 (Incompleteness) Let Nat be a categorical specification of the Booleans and the natural numbers that axiomatizes the constants

$$\begin{aligned} 0, 1 &: B \\ \sigma &: N \to N \\ o &: N \end{aligned}$$

with their usual meaning. Find A and e such that Nat, $A \vDash e$ and Nat, $A \nvdash e$.

Hints: Choose an infinite A and exploit the compactness of deductive entailment. Use a constant $f\colon N \to B$ that does not appear in Nat. Choose e such that it requires f to represent an infinite set.

Exercise 9.5 (Termination) Let PL be a specification that axiomatizes the constants

$$\begin{aligned} 0, 1 &: B \\ \wedge, \to &: B \to B \to B \\ \forall, \exists &: (V \to B) \to B \end{aligned}$$

with their usual meaning once V is fixed.

a) Let $r : V \to V \to B$ be a constant that doesn't occur in PL. Find an equation ter such that a model of PL is a model of PL∪{ter} if and only if it interprets r with a terminating relation.

b) Find equations A such that

   (i) PL,ter,A $\models 0 = 1$

   (ii) PL,ter,A $\not\models 0 = 1$

   Hint: Use pairwise distinct constants $a_1, a_2, \ldots$ that don't occur in PL and exploit the compactness of deductive entailment.

Exercise 9.6 (Finiteness) Let PL be a specification that axiomatizes the constants

$$\begin{aligned}
0, 1 &: B \\
\to &: B \to B \to B \\
\forall, \exists &: (V \to B) \to B \\
\doteq &: V \to V \to B
\end{aligned}$$

with their usual meaning once V is fixed.

a) Find an equation fin such that a model of PL is a model of PL∪{fin} if and only if it interprets V with a finite set.

   Hint: A set X is finite if and only if every injective function $X \to X$ is surjective.

b) Find equations A such that

   (i) PL,fin,A $\models 0 = 1$

   (ii) PL,fin,A $\not\models 0 = 1$

Exercise 9.7 (Transitive Closure) Let PL be a specification that axiomatizes the constants

$$\begin{aligned}
0, 1 &: B \\
\wedge, \vee, \to &: B \to B \to B \\
\forall, \exists &: (V \to B) \to B \\
\forall &: ((V \to V \to B) \to B) \to B
\end{aligned}$$

with their usual meaning once V is fixed. We use the notation $Rel := V \to V \to B$.

a) Give a term trans of type $Rel \to B$ that describes a function that tests whether a relation is transitive.

b) Give a term incl of type $Rel \to Rel \to B$ that describes a function that tests

$r \subseteq r'$.

c) Give a term tc of type $Rel \to Rel \to B$ that describes a function that tests whether the second argument is the transitive closure of the first argument.

d) Let $r, r' : Rel$ be constants that don't occur in PL. Show that

$$PL \cup \{r'xy = rxy \vee (\exists z.rxz \wedge r'zy)\}$$

has models where r' is not the transitive closure of r.

Exercise 9.8 Draw a locally confluent relation that is not confluent.

Hint: Two non-terminal and two terminal nodes suffice.

Exercise 9.9 Draw an infinite relation $\to \subseteq \mathbb{N}^2$ that is confluent, non-terminating, and where every $n \in \mathbb{N}$ has exactly one normal form.

Exercise 9.10 Draw a terminating relation $\to$ such that $\exists x\ \forall n \in \mathbb{N}\ \exists y\colon\ x \to^n y$.

Exercise 9.11 Make sure that you understand the proofs of the following facts.

a) $\to$ semi-confluent $\implies$ $\to$ Church-Rosser

b) $\to$ terminating and locally confluent $\implies$ $\to$ confluent

Exercise 9.12 Let $\to\ \subseteq X^2$ be confluent and terminating. Prove the following statements:

a) There is no $x \in X$ that has two normal forms.

b) $\forall x, y \in X\colon\ x \leftrightarrow^* y \implies x, y$ have the same normal forms.

You may use the following facts:

i) $\to$ confluent $\implies \to$ Church-Rosser

ii) $\to$ terminating $\implies \forall x \in X\colon\ x$ has a normal form