Assignment 1
Introduction to Computational Logic, SS 2006

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Exercise 1.1 (Boolean Connectives) Consider the values

$$0, 1 \in \mathbb{B}$$
$$\neg \in \mathbb{B} \to \mathbb{B}$$
$$\wedge, \vee, \Rightarrow, \Leftrightarrow \in \mathbb{B} \to \mathbb{B} \to \mathbb{B}$$

a) Describe 1, $\neg$, $\wedge$, $\vee$, $\Leftrightarrow$ with 0, $\Rightarrow$, and λ.
b) Describe 0, $\wedge$, and $\Rightarrow$ with 1, $\neg$, $\vee$, and λ.

Exercise 1.2 (Sets as functions) Let X be a set. The subsets of X can be represented as the functions $X \to \mathbb{B}$. We use $P(X)$ as abbreviation for $X \to \mathbb{B}$. Express the following set operations with the logical operations $\neg$, $\wedge$, $\vee$, $\forall_X$ and λ.
a) Intersection $\cap \in P(X) \to P(X) \to P(X)$
b) Union $\cup \in P(X) \to P(X) \to P(X)$
c) Difference $- \in P(X) \to P(X) \to P(X)$
d) Subset $\subseteq \in P(X) \to P(X) \to \mathbb{B}$
e) Disjointness $\| \in P(X) \to P(X) \to \mathbb{B}$
f) Membership $(\in) \in X \to P(X) \to \mathbb{B}$

Exercise 1.3 (Identities and Quantifiers) Express
a) $\forall_X$ with $\doteq_{X \to \mathbb{B}}$.
b) $\exists_X$ with $\forall_X$ and $\neg$.
c) $\forall_X$ with $\exists_X$ and $\neg$.
d) $\doteq_{X \to Y}$ with $\forall_X$ and $\doteq_Y$.
e) $\doteq_{\mathbb{B}}$ with $\Leftrightarrow$.
f) $\doteq_X$ with $\forall_{X \to \mathbb{B}}$ and $\Rightarrow$. Challenge! Start with $\forall_{X \to \mathbb{B}}$ and $\Leftrightarrow$.

Exercise 1.4 (Tree Representatations) Draw the tree representations of the following terms. Represent bound variables with de Bruijn indices.
a) $\lambda x : \mathbb{B}.\ \lambda y : \mathbb{B}.\ \neg x \vee y$
b) $\lambda x : X.\ f(\lambda y : Y.\ gyx)xy$
c) $(\forall x : X\ .\ fx \wedge gx) \Leftrightarrow \forall f \wedge \forall g$

Exercise 1.5 (Substitution) Apply the following substitutions:

a) $(fx(\lambda x : X.\, y))[x := y]$
b) $(\lambda x : X.\ fxy)[y := x]$
c) $(\lambda x : X.\ \lambda y : X.\ fxy)[f := gxy]$

Exercise 1.6 (Term Equality) The following notations describe terms. Draw for each notation the tree representation of the term described. Represent bound variables with de Bruijn indices. Which of the notations describe the same term? (Two terms are equal if they have the same tree representation.)

a) $\lambda x : X.\ x$
b) $\lambda y : X.\ y$
c) $\lambda y : X.\ x$
d) $(\lambda x : X.\ x)[x := y]$
e) $(\lambda x : X.\ y)[y := x]$
f) $\lambda x : X.\ \lambda y : X.\ fxy$
g) $\lambda y : X.\ \lambda x : X.\ fyx$