CS 578 – Cryptography

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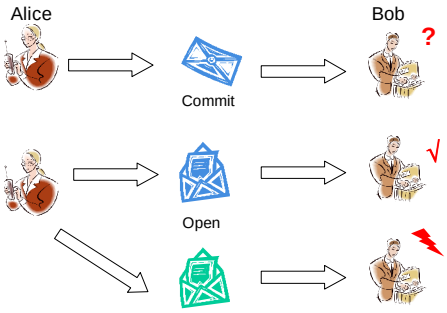
Commitment Schemes, Secret Sharing

July 7, 2006

Administrative Announcements

- Handouts
  - "Secret Sharing" on Tuesday

Recall: Commitment Schemes



Recall: Commitment Schemes

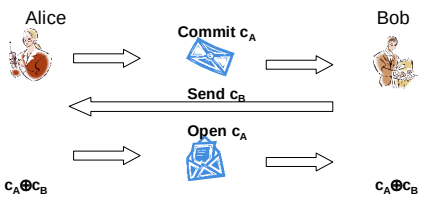
- Definition (Commitment Scheme): A (two-party) commitment scheme consists of two interactive algorithms (commit, open):
  - commit: The committer receives a message m; the recipient obtains no input. Neither committer nor recipient make any output except for a value acc ∈ {ok, error}.
  - open: The same committer and recipient (maintaining state from the commit phase) take part. No inputs are needed. Recipient either outputs “(accept, m)” for some m or “reject”.

Recall: Commitment Schemes

- Properties of a commitment scheme
  - Correctness: If both committer and recipient are honest and run *commit* (on m) and then *open*, then the recipient outputs (accept, m).
  - Binding: Even if the committer is dishonest, he cannot open the commitment in two different ways, i.e., after a successful *commit*, causing the recipient to output (accept, m) or (accept, m') in the *open* protocol.
  - Hiding: The *commit* protocol should not give the recipient any information on m.

Application: Coin Tossing

- Idea: Alice and Bob throw a coin c_A, c_B and compute $c_A \oplus c_B$.
- But: c_A, c_B may be correlated!



Recall: An Impossibility Result

- Theorem: No commitment scheme can be information-theoretically binding and information-theoretically hiding.
- (But information-theoretically binding and computationally hiding is possible, and vice versa)

Recall: The Quadratic Residuosity Scheme

- Quadratic residuosity scheme (for single bits)
- *commit* (on m):
  - Committer randomly chooses n-bit primes p,q with p=3 mod 4 and q = 3 mod 4, and sets N:=pq.
    Committer then chooses $y \leftarrow_R Z^*_N$, computes
    $$x := (-1)^m \cdot y^2 \bmod N$$
    and outputs com = (N,x).
- *Open*:
  - Committer sends (m, p, q, y).
  - Recipient checks if p,q primes with p=3 mod 4 and q = 3 mod 4, and if N=pq. Then he checks if $x = (-1)^m \cdot y^2 \bmod N$. If so, he outputs (accept, m), else reject.

Recall: The Quadratic Residuosity Scheme

- Theorem: The quadratic residuosity commitment scheme is information-theoretically binding, and it is computationally hiding under the quadratic residuosity assumption.

  [proof on the board]
- Extension to multiple bits easily doable

Information-theoretically Hiding Schemes

- Discrete Logarithm scheme (for messages from G_q)
- *commit* (on m):
  - Recipient picks random n-bit prime q, random $n_p(n)$-bit prime p such that q | p-1 and two random elements g,h ∈ Z^*_p of order q. Recipient sends (p,q,g,h) to the committer.
  - Committer verifies that p and q are prime and that g and h are of order q.
  - Committer then chooses $k \leftarrow_R \{1,\ldots,q\}$ and outputs
    $com := g^m h^k \bmod p$
- *Open*:
  - Committer sends (m, k).
  - Recipient verifies that $com = g^m h^k \bmod p$. If so, he outputs (accept, m), else reject

The Discrete-Logarithm Scheme

- Theorem: The discrete logarithm commitment scheme is information-theoretically hiding, and it is computationally binding under the Discrete Logarithm assumption for subgroups G_q of Z^*_p.

  [proof on the board]

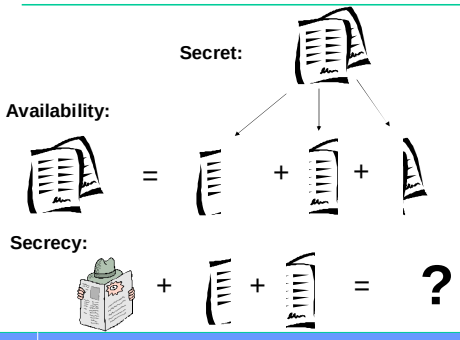
- Very efficient scheme:
  - Commit only requires two exponentiations with exponent of size log q
  - Commitment only one element of size log p

The Discrete-Logarithm Scheme

- Special properties of the scheme
  - The discrete logarithm commitment scheme is homomorphic if several values are committed using the same value pk_{REC} = (p,q,g,h):
    If a committer has made two commitments com_1 and com_2 with respect to the same value pk_{REC} = (p,q,g,h), he can open the product $com_1 \cdot com_2$ as a commitment to m_1+m_2 mod q.
    Opening the product commitment does not reveal additional information about the individual contents.

  [proof on the board]

Secret Sharing



Coca-Cola trade secrets 'stolen'

US prosecutors have charged three people with stealing secrets from soft drinks company Coca-Cola and trying to sell them to its main rival PepsiCo.

[Coca Cola's] chief executive, Neville Isdell, said that "information is the lifeblood of the company".

Mr Isdell said that Coca-Cola would be reviewing its security procedures.

www.bbc.co.uk, 7/6/06

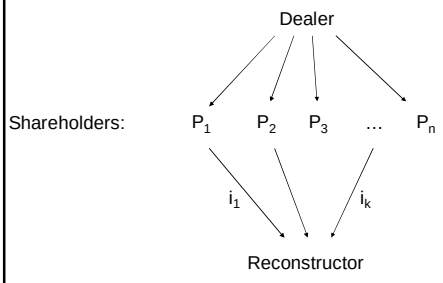
Police strike at heart of mafia

A bloody mafia power struggle appears to have been narrowly averted after dawn raids across the Sicilian capital of Palermo on Tuesday netted 24 alleged Cosa Nostra "godfathers", including a mobster said by investigators to be trying to take over as the next capo di tutti i capi, or "boss of all bosses".

"It came two months after the seizure of the crime syndicate's fugitive commander, **Bernardo Provenzano**, **and was made possible, in part, by evidence found at his hideout**."

www.guardian.co.uk, 6/20/2006

Secret Sharing



Definition k-of-n Secret Sharing

- Participants:
  - Dealer, Reconstructor, (n) Shareholders $P_1,\ldots,P_n$
- Algorithms
  - a set of secrets S
  - Share(s) = $s_1,\ldots,s_n \in S_i$
  - Reconstruct($s_{i1},\ldots,s_{ik}$) = s*
- Requirements
  - **Availability**: s*=s if dealer, reconstructor, and the k shareholders are honest.
  - **Secrecy**: An adversary learning at most k values s_{ij} gains *no* information (information theoretical) about the secret s

Simple Examples

- Bad example:
```
pas-----
---swo--
------rd
```
- Good Example (n-of-n)
  - To share bit b:

    $r_1, r_2 \leftarrow_R \{0,1\}$

    $r_3 := b \oplus r_1 \oplus r_2$
  - Shares: r_1, r_2, r_3

XOR-scheme

- Availability:

  $r_1 \oplus r_2 \oplus r_3 = r_1 \oplus r_2 \oplus (b \oplus r_1 \oplus r_2) = b$

- Secrecy: Given any two shares s_1, s_2, this can be opened to *any* secret b:

  $s_1 \oplus s_2 \oplus (s_1 \oplus s_2 \oplus 0) = 0$

  $s_1 \oplus s_2 \oplus (s_1 \oplus s_2 \oplus 1) = 1$

Algebraic Fact

- **Fact:** Let p be a polynomial of degree k-1 over a field F. If p has k zero points, then $p \equiv 0$.
- **Corollary:** Let pol and pol' be polynomials of degree k-1 that coincide at k points. Then these polynomials are equal.

Proof: Let $x_1,\ldots,x_k$ such that

$pol(x_i) = pol'(x_i) \Rightarrow (pol - pol')(x_i)=0$

$\Rightarrow (pol - pol') \equiv 0$

$\Rightarrow pol \equiv pol'$

Shamir's scheme (k-of-n)

- Parameters:
  - A finite field F where $|F|>n$, and a set $S \subseteq F$ of secrets, for simplicity $S = F = Z_p$
- For each participant i, fix

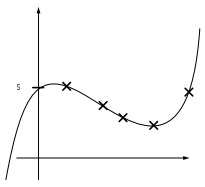
  $0 \neq x_i \in Z_p$

  such that

  $x_i \neq x_j$ for $i \neq j$

  (publicly known), e.g.,

  $x_i = i$



Shamir: Sharing

- Sharing:
  - choose random polynomial of degree k-1 (over Z_p), i.e.,

    $a_i \leftarrow_R Z_p$ (i=1,…,k-1),

    $pol(x) = a_{k-1}x^{k-1} + \ldots + a_1x^1 + s$
  - Compute

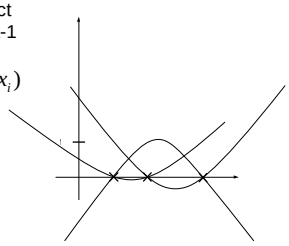
    $y_i := pol(x_i)$
  - Give participant i the value y_i

Shamir: Reconstruction

- Let Avail=$\{i_1,\ldots,i_k\}$ the set of indices of available shares.
- For all $i^* \in$ Avail, construct polynomials of degree k-1

$$cpol_{i^*}(x) := \prod_{i \in Avail \setminus \{i^*\}} (x - x_i)$$

$$bpol_{i^*}(x) := \frac{cpol_{i^*}(x)}{cpol_{i^*}(x_{i^*})}$$



Shamir: Availability

- Let

  $pol^*(x) := y_{i1}\, bpol_{i1}(x) + \ldots + y_{ik}\, bpol_{ik}(x)$
- For all $i \in \{i_1,\ldots,i_k\}$ we have

  $pol^*(x_i) = y_i$

  [proof on the board]
- A pol and pol* are of degree k-1 and coincide in k point, thus

  $pol(x) = pol^*(x)$,

  in particular

  $s = pol(0) = pol^*(0) = s^*$

Shamir: Secrecy

- (Information-theoretical) secrecy:
  For all k-1 indices $i_1,\ldots,i_{k-1}$,
  for all shares $s_{i1},\ldots,s_{ik-1,}$ and
  for all secrets s,
  there exists a unique polynomial pol* which is consistent with these values

  [proof on the board]

Shamir: Improvements

- Reconstruction: No need to reconstruct the whole polynomial:

$$s = \sum_{i \in Avail} y_i \cdot bpol(0)$$

where

$$bpol_i(0) = \prod_{i \in Avail \setminus \{i*\}} \frac{-x_i}{x_{i*} - x_i}$$

Shamir: Improvements (cont'd)

- Splitting long secrets:
  - complexity is quadratic in the size of the secret
  - Splitting the secret into parts of constant size
    $\Rightarrow$ linear complexity
- Sharing several secrets:
  - The polynomials $bpol_i$ can be re-used

Shamir: Optimality

- Shares are as big as the secret, can one do better?

  No!
- One can proof that this is optimal, i.e.,

  $|S| \leq |S_i|$

  [Proof sketch]

Shamir: Computing with Secrets

- Let two shares be given

  $s=pol(0)$ and $s'=pol'(0)$,

  $y_i=pol(x_i)$ and $y'_i=pol'(x_i)$
- Then the $y_i+y'_i$ are shares for $s+s'$

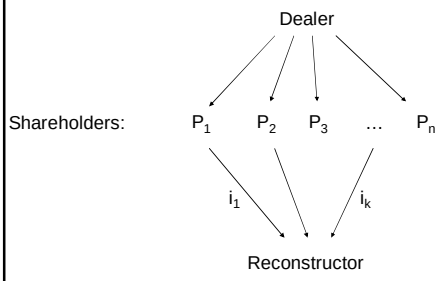
  [proof on the board]
- For $c \in Z_p$: The $c \cdot y_i$ are shares for $c \cdot s$

  [proof on the board]

Proactive Secret Sharing

- If one share gets compromised?
- Not practical to change the secret, so the uncompromised shares need to be renewed.
- Dealer generates a new random polynomial with constant term zero
- Computes a new $y'_i:=pol^*(x_i)$, shareholder adds y'_i and his y_i
- Attacker can only recover the secret if he can find enough other non-updated shares to reach the threshold.

Verifiable Secret Sharing



Verifiable Secret Sharing (VSS)

- Malicious shareholders: If one uses a wrong share, the secret cannot be reconstructed
- Malicious dealer: Can output inconsistent shares

⇒ Pedersen's scheme

Pedersen: Sharing

- Create a Shamir share (a_0=s):
  $pol(x) = a_{k-1}x^{k-1} + \ldots + a_1x^1 + a_0$,
  y_i:=$pol(x_i)$
- The dealer commits to the a_i using the DLog scheme, i.e.,
  $b_i \leftarrow_R Z_p$, $com_i = g^{a_i}h^{b_i}$ (i=0,…,n),
  these commitments are broadcasted to everyone.
- Let
  $pol' := b_{k-1}x^{k-1} + \ldots + b_1x^1 + b_0$
  z_i:=$pol'(x_i)$
- Shareholder i gets y_i, z_i

Pedersen: Sharing (cont'd)

- Shareholder verifies that

$$g^{y_i} h^{z_i} = com_{k-1}^{x_i^{k-1}} \cdot \ldots \cdot com_1^{x_i^1} \cdot com_0 =: comshare_i$$

[proof on the board]

Pedersen: Reconstruction

- The reconstructor tests if

$$g^{y_i} h^{z_i} = comshare_i$$

- If this is correct, then reconstruct the secret from the y_i's as before