CS 578 – Cryptography

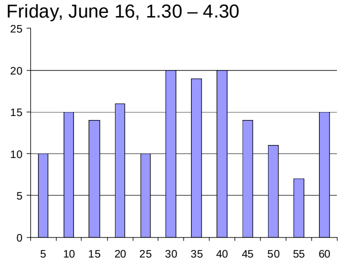
Prof. Michael Backes

The Cramer-Shoup Encryption Scheme - Security Proof -

June 9, 2006

Administrative Announcements

- Time for investigating your exam:
  - Friday, June 16, 1.30 – 4.30



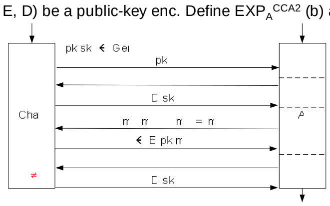
Recall: Public-key Encryption

- Definition (Public-Key Encryption Scheme): A public-key encryption scheme is a triple of efficient algorithms (Gen,E,D):
  - Gen(n): Generates a secret/public key pair (pk,sk) for security parameter n
  - E(pk,m) and D(sk,m) as usual

  such that for all (pk,sk) ← Gen(k), and for all m: D(sk,E(pk,m)) = m
- n is the security parameter, tacitly considered input to all algorithms (formally again sequences of encryption schemes and (uniform) adversaries, but much more natural for public-key encryption).

Definition of CCA2

- Let (Gen, E, D) be a public-key enc. Define EXP_A^{CCA2} (b) as:



- Definition (Semantic Security against CCA2). PE = (Gen, E, D) is semantically secure under chosen-ciphertext attack (CCA2) if for all efficient adversaries A, the following is negligible: $Adv^{CCA2}[A,PE] = |Pr[EXP_A^{CCA2}(0)=1] - Pr[EXP_A^{CCA2}(1)=1]|$.

Keyed Hash Functions

- Let $Hash = (H(pk, \cdot))_{pk \in [Gen(n)]}$ be a keyed family of hash functions, i.e., $H(pk, \cdot): \mathcal{M}_{pk} \rightarrow \mathcal{T}_{pk}$ (usually $\{0,1\}^* \rightarrow \{0,1\}^{l(n)}$)

- Definition (Collision-resistance for family of (keyed) hash functions): A family *Hash* of keyed hash functions is collision-resistant if for all efficient adversaries A (in security parameter n), we have that

  $Pr[\ H(pk,m) = H(pk,m') \wedge m \neq m'\ ;\ pk \leftarrow Gen(n), (m,m') \leftarrow A(n,pk)\ ]$

  is negligible (in n).

The Cramer-Shoup Encryption Scheme

- Key generation for security parameter n:
  - Pick random n-bit prime q
  - Pick random $n_p(n)$-bit prime p such that q | p-1
  - Pick $g_1 \in Z^*_p$ of order q and second generator g_2 of $\langle g_1 \rangle$ randomly
  - Pick random $x_1, x_2, y_1, y_2, z \in \{1,\ldots,q\}$
  - Set

    $$s = g_1^{x_1} \cdot g_2^{x_2} \qquad t = g_1^{y_1} \cdot g_2^{y_2} \qquad h = g_1^{z}$$

  - Let $pk_{hash} \leftarrow Gen_{Hash}(n)$ (Denote $H(\cdot) := H(pk_{hash}, \cdot)$)
  - Set pk := $(q, p, g_1, g_2, s, t, h, pk_{hash})$
  - Set sk := $(pk, x_1, x_2, y_1, y_2, z)$

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  - Pick random x_1, x_2, y_1, y_2, $z \in \{1,\ldots,q\}$
  - Set

    $$s = g_1^{x_1} \cdot g_2^{x_2} \qquad t = g_1^{y_1} \cdot g_2^{y_2} \qquad h = g_1^z$$
  - Let $pk_{hash} \leftarrow Gen_{Hash}(n)$ (Denote $H(\cdot) := \mathcal{H}(pk_{hash}, \cdot)$)
  - Set pk := (q, p, g_1, g_2, s, t, h, pk_{hash})
  - Set sk := (pk, x_1, x_2, y_1, y_2, z)

The Cramer-Shoup Encryption Scheme

- Encryption Enc(pk,m) where $m \in \langle g_1 \rangle = G_q$ and pk = (q, p, g_1, g_2, s, t, h, pk_{hash})
  - Pick r randomly from {1,…,q}
  - Set

    $$i_1 = g_1^r \qquad c^* = h^r \cdot m$$
  - In addition, set

    $$i_2 = g_2^r \qquad \alpha = H(i_1, i_2, c^*) \qquad v = s^r \cdot t^{r\alpha}$$
- The ciphertext is

  $$c = (i_1, i_2, c^*, v)$$

The Cramer-Shoup Encryption Scheme

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The Cramer-Shoup Encryption Scheme

- Decryption Dec(sk,c) where c = (i_1, i_2, c*, v) and sk = (pk, x_1, x_2, y_1, y_2, z)
  - Compute
    $$\alpha = H(i_1, i_2, c^*)$$
  - Verify if the following holds. If not abort.
    $$i_1^{x_1+y_1\alpha} \cdot i_2^{x_2+y_2\alpha} = v$$
  - If verification is true, compute
    $$k = i_1^z \qquad m = \frac{c^*}{k}$$

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The Cramer-Shoup Encryption Scheme

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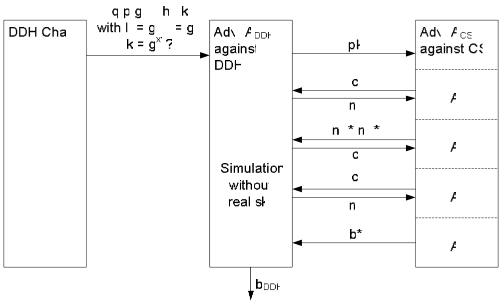
The Cramer-Shoup Encryption Scheme

- Correctness of decryption, resistance against naïve ElGamal attacks shown last time
- Intuition on why the test (using v) rejects “misformed” ciphertexts
  - If $i_2 = g_2^r$ then v necessarily of the correct form
  - If $i_2 \neq g_2^r$ then some suitable value
    $$v := i_1^{x_1+y_1\alpha} \cdot i_2^{x_2+y_2\alpha}$$
    exists, but we will show that an attacker not knowing the secret key can only find such a v with negligible probability.

High-level Overview of the Reduction



Intuition on the Reduction

- Basic idea: Use given DH triple (h,i,k) (which is either (g^x, g^y, g^{xy}) or (g^x, g^y, g^z)), in public key and challenge-ciphertext such that:
  - We get a correct simulation if $k=g^{xy}$ → success probability of A_{CS} and hence of A_{DDH} significantly better than ½.
  - The adversary A_{CS} learns no information about b if $k=g^z$, i.e., if k random → success probability of A_{CS} and hence of A_{DDH} = ½.

→ Difference not negligible → DDH broken

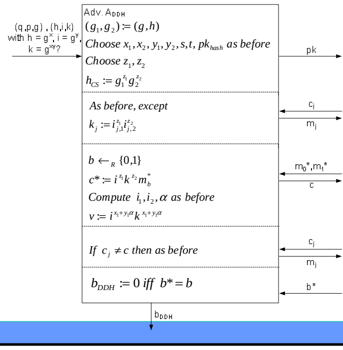
Intuition on the Reduction (cont'd)

- Where to use the DH triple?
- Set

$$g_1 := g \qquad (g_2, i_1, i_2) := (h, i, k)$$

- If $k=g^{xy}$ then i_1,i_2 chosen according to correct probability distribution
- Problem: A_{DDH} does not know y and hence has to compute remaining components of the ciphertext differently.

Detailed Overview of the Reduction



Simulation of Key Generation

- Given (p,q,g), (h,i,k)
- Use $g_1 := g$ and $g_2 := h$
- Pick random x_1, x_2, y_1, y_2, z_1, z_2 $\in \{1,\ldots,q\}$
- Set

$$s = g_1^{x_1} \cdot g_2^{x_2} \qquad t = g_1^{y_1} \cdot g_2^{y_2} \qquad h_{CS} = g_1^{z_1} g_2^{z_2}$$

- Let $pk_{hash} \leftarrow Gen_{Hash}(n)$ (Denote $H(\cdot) := H(pk_{hash}, \cdot)$)
- Output pk := (q, p, g_1, g_2, s, t, h, pk_{hash})
- Remark: Only cheated with h_{CS} so far. But ok since suitable z can be defined as

$$z = z_1 + xz_2$$

Simulation of Ciphertext Decryption

- Given c = ($i_{j,1}$, $i_{j,2}$, c_j^*, v_j)
- A_{DDH} knows entire secret key, but "wrong" z and h_{CS}
- Verification of v as usual (uses neither z nor h_{CS})
- Usually, one would compute $i_{j,1}^z$, but z unknown to A_{DDH}.
- Let $i_{j,1} = g_1^{r_j}$ (but A_{DDH} does not know this r_j !). Then

$$k_j = i_{j,1}^z = g_1^{r_j z} = (g_1^{z_1} \cdot g_2^{z_2})^{r_j} = g_1^{r_j z_1} \cdot g_2^{r_j z_2}$$

- If $i_{j,2} = g_2^{r_j}$ (correct ciphertext) then

$$k_j = i_{j,1}^{z_1} \cdot i_{j,2}^{z_2}$$

- Return $m_j = c^*_j / k_j$
- Does not work for incorrect ciphertexts!
- Later show lemma: Adversary only finds correct v's for incorrect (misformed) ciphertexts with negligible probability.

Simulation of Encryption Challenge

- Given ($h = g^x$, $i = g^y$, $k = g^{xy}$?) and (m_0, m_1)
- Choose bit b randomly and set $i_1 := i$ and $i_2 := k$.
- For computing $c^* = h_{CS}{}^y \cdot m_b$, A_{DDH} had to know y, but doesn't.
- Instead, it computes

$$k^* := h_{CS}^y = (g_1^{z_1} \cdot g_2^{z_2})^y = g_1^{yz_1} \cdot g_2^{yz_2} = i^{z_1} \cdot k^{z_2} \qquad c^* = i^{z_1} \cdot k^{z_2} \cdot m_b$$

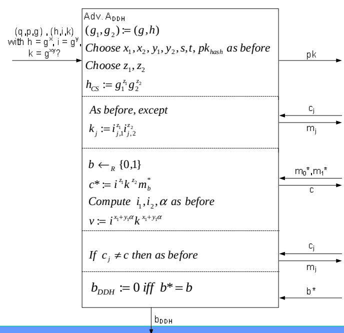
- v again hard to compute because y unknown. Instead compute

$$v = i_1^{x_1 + y_1 \alpha} \cdot i_2^{x_2 + y_2 \alpha}$$

i.e., as one would in the decryption routine.

- Perfectly correct simulation if $k = g^{xy}$ although y unknown to A_{DDH} (always found alternative ways to compute the needed values)

Detailed Overview of the Reduction



Correct Simulation if $k = g^{xy}$

- Shown correct simulation in case $k = g^{xy}$ except for decryption of misformed ciphertexts
- Lemma: If $k = g^{xy}$, the probability that A_{CS} succeeds in sending any ciphertext c_j whose first two components are not of the form

$$i_{j,1} = g_1^{r_j} \qquad i_{j,2} = g_2^{r_j}$$

for some r_j, but which nevertheless passes the verification, is exponentially small.

- Lemma holds even for information-theoretic adversaries (easier to prove)

Correct Simulation if $k = g^{xy}$

- Fix a ciphertext ($i_{j,1}$, $i_{j,2}$, c_j^*, v_j) and assume

$$i_{j,1} = g_1^{r_j} \qquad i_{j,2} = g_2^{r_j^*}$$

  for some $r_j \neq r^*_j$ mod q.
- This ciphertext passes verification if

$$v_j = i_{j,1}^{x_1 + y_1\alpha_j} \cdot i_{j,2}^{x_2 + y_2\alpha_j} = g_1^{r_j(x_1 + y_1\alpha_j)} \cdot g_2^{r_j^*(x_2 + y_2\alpha_j)}$$

- Let β_j such that $v_j = g_1^{\beta j}$ .Then verification is true if and only if (mod q)

$$\beta_j = r_j(x_1 + y_1\alpha_j) + xr_j^*(x_2 + y_2\alpha_j)$$

- Secrets here are x_1, x_2, y_1, y_2.

Correct Simulation if $k = g^{xy}$

- If x_1, x_2, y_1, y_2 were perfectly secret
  → clear that A_{CS} gets no information about β_j
  → cannot construct v_j except for purely guessing it
- But A_{CS} gets information on x_1, x_2, y_1, y_2 :
  - 1. Key generation: s and t contain information

$$s = g_1^{x_1} \cdot g_2^{x_2} \qquad t = g_1^{y_1} \cdot g_2^{y_2}$$

    yielding equations

$$\sigma = x_1 + xx_2 \qquad \tau = y_1 + xy_2$$

  - 2. Decryption of correct ciphertexts: Attacker only learns

$$k_j = i_{j,1}^{z_1} \cdot i_{j,2}^{z_2}$$

    which is independent of x_1, x_2, y_1, y_2

Correct Simulation if $k = g^{xy}$

- But A_{CS} gets information on x_1, x_2, y_1, y_2 (cont'd):
  - 3. Challenge encryption: Here c* only depends on z_1, z_2, but

$$v = i_1^{x_1 + y_1\alpha} \cdot k^{x_2 + y_2\alpha}$$

    depends on x_1, x_2, y_1, y_2.
    However this gives no new info since

$$v = s^r \cdot t^{r\alpha}$$

    in this correct case, and s and t are known already.
  - 4. Answers on incorrect ciphertexts: The case we are currently treating: With overwhelming probability, the answer will be the fixed error message.

→ Investigate linear equations to see what the adversary has learned (prob. ≈ 1/q of guessing β_j right)

The case k random

- Show that m_b perfectly hidden by $k^* = i^{z_1} \cdot k^{z_2}$ (in a OTP manner)
- Problem again: If k* perfectly secret, this would be clear, but A_{CS} gets information on k* via z_1, z_2:
  - 1. Key generation: h_{CS}:

    $$h_{CS} = g_1^{z_1} g_2^{z_2}$$

    yielding an equation

    $$z = z_1 + x z_2$$
  - 2. Decryption of correct ciphertexts: Attacker only learns

    $$k_j = i_{j,1}^{z_1} \cdot i_{j,2}^{z_2} = (g_1^{z_1} g_2^{z_2})^{r_j} = h_{CS}^{r_j}$$

    and thus no new info on z_1, z_2.

The case k random

- But A_{CS} gets information on z_1, z_2 (cont'd):
  - 3. Challenge encryption: This is the case we are just considering
  - 4. Answers on incorrect ciphertexts: Show below: With overwhelming probability, the answer is the fixed error message.
- In summary, the attacker learns at most one linear equation about the two variables z_1, z_2.
  → q pairs (z_1, z_2) still possible for the adversary
- The equation $k^* = i^{z_1} \cdot k^{z_2}$ can be written as

  $$\gamma = y z_1 + \delta z_2$$

  for $k^* = g_1^{\gamma}$ and $k = g_1^{\delta}$
- Linearly independent of above eq. (except if $k = g^{xy}$)

The case k random

- Final lemma: If $k = g^z$, i.e., k random, the probability that A_{CS} succeeds in sending any ciphertext c_j whose first two components are not of the form

  $$i_{j,1} = g_1^{r_j} \qquad i_{j,2} = g_2^{r_j}$$

  for some r_j, but which nevertheless passes the verification, is exponentially small.
- Lemma holds again for information-theoretic adversaries
- Proof very similar to lemma for $k = g^{xy}$

The case k random

- Fix a ciphertext ($i_{j,1}$, $i_{j,2}$, c_j*, v_j) and assume
  $$i_{j,1} = g_1^{r_j} \qquad i_{j,2} = g_2^{r_j^*}$$
  for some $r_j \neq r^*_j$ mod q.
- This ciphertext passes verification if and only if
  $$\beta_j = r_j(x_1 + y_1\alpha_j) + xr_j^*(x_2 + y_2\alpha_j)$$
  where β_j such that $v_j = g_1^{\beta j}$
- Secrets here again x_1, x_2, y_1, y_2.

The case k random

- If x_1, x_2, y_1, y_2 were perfectly secret
  → clear that A_{CS} gets no information about β_j
  → cannot construct v_j
- But A_{CS} gets information on x_1, x_2, y_1, y_2 :
  - 1. Key generation: s and t
    $$s = g_1^{x_1} \cdot g_2^{x_2} \qquad t = g_1^{y_1} \cdot g_2^{y_2}$$
    yielding equations
    $$\sigma = x_1 + xx_2 \qquad \tau = y_1 + xy_2$$
  - 2. Decryption of correct ciphertexts: Attacker only learns
    $$k_j = i_{j,1}^{z_1} \cdot i_{j,2}^{z_2}$$
    which is independent of x_1, x_2, y_1, y_2

The case k random

- But A_{CS} gets information on x_1, x_2, y_1, y_2 (cont'd):
  - 3. Challenge encryption (Difference to last lemma!!): Here c* depends only on z_1, z_2, but
    $$v = i_1^{x_1 + y_1\alpha} \cdot k^{x_2 + y_2\alpha}$$
    depends on x_1, x_2, y_1, y_2. In the last lemma, this was
    $$v = s^r \cdot t^{r\alpha}$$
    for known s and t. In this case, we get an additional equation:
    $$\varepsilon = y(x_1 + y_1\alpha) + \delta(x_2 + y_2\alpha)$$
    where $k = g_1^{\delta}$ and $v = g_1^{\varepsilon}$.

The case k random

- But A_{CS} gets information on them (cont'd):
  - 4. Answers on incorrect ciphertexts: The case we are currently treating: With overwhelming probability, the answer will be the fixed error message.
- Finally (and then we are done):

  Investigate linear dependency of equations to see what the adversary has learned
  - 1. Case: If $\alpha \neq \alpha_j$ and $k \neq g^{xy}$
    → q tuples possible from the adversaries point of view, but only one is correct (prob. 1/q of guessing β_j right)
  - 2. Case: $k = g^{xy}$ → Exponentially small probability.
  - 3. Case: $\alpha = \alpha_j$ → Collision found (or guessed the challenge ciphertext before the challenge phase)!