Computer Architecture 1 - Übungsblatt 1

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Aufgabe 1: partition lemma

$$\begin{aligned}
\langle a\,[n-1:m]\rangle \cdot 2^m + \langle a\,[m-1:0]\rangle &\overset{\text{Def.}}{=} 2^m \cdot \sum_{i=m}^{n-1} a_i \cdot 2^{i-m} + \sum_{i=0}^{m-1} a_i \cdot 2^i \\
&= \sum_{i=m}^{n-1} (a_i \cdot 2^{i-m} \cdot 2^m) + \sum_{i=0}^{m-1} a_i \cdot 2^i \\
&= \sum_{i=m}^{n-1} a_i \cdot 2^i + \sum_{i=0}^{m-1} a_i \cdot 2^i \\
&= \sum_{i=0}^{n-1} a_i \cdot 2^i \\
&\overset{\text{Def.}}{=} \langle a\,[n-1:0]\rangle
\end{aligned}$$

$\square$

Aufgabe 2: binary fractions

$$
\begin{aligned}
\langle a[n-1:0].f[1:p]\rangle &= \sum_{i=0}^{n-1} a_i \cdot 2^i + \sum_{i=1}^{p} f_i \cdot 2^{-i} \\
&= \sum_{i=p}^{p+n-1} a_{i-p} \cdot 2^{(i-p)} + \sum_{i=1}^{p} f_i \cdot 2^{-i} \\
&= \sum_{i=p}^{p+n-1} a_{i-p} \cdot 2^i \cdot 2^{-p} + \sum_{i=1}^{p} f_i \cdot 2^{-i} \\
&= \left(\sum_{i=p}^{p+n-1} a_{i-p} \cdot 2^i + \sum_{i=1}^{p} f_i \cdot 2^{-i} \cdot 2^p \right) \cdot 2^{-p} \\
&= \left(\sum_{i=p}^{p+n-1} a_{i-p} \cdot 2^i + \sum_{i=1}^{p} f_i \cdot 2^{p-i} \right) \cdot 2^{-p} \\
&= \left(\sum_{i=p}^{p+n-1} a_{i-p} \cdot 2^i + \sum_{i=0}^{p-1} f_{p-i} \cdot 2^i \right) \cdot 2^{-p} \\
&= \langle af \rangle \cdot 2^{-p}
\end{aligned}
$$

Aufgabe 3: basic properties of two's complement numbers

- sign extension: $[a] = [a_{n-1}a]$

$$\begin{aligned}[a_{n-1}a] &= -a_{n-1} \cdot 2^n + \langle a\,[n-1:0]\rangle \\ &= -a_{n-1} \cdot 2^n + a_{n-1} \cdot 2^{n-1} + \langle a\,[n-2:0]\rangle \\ &= [a]\end{aligned}$$

□

- sign bit: $[a] < 0 \leftrightarrow a_{n-1} = 1$
  zu zeigen: $\langle a[n-2:0]\rangle < 2^{n-1}$

$$\begin{aligned}[a] < 0 \Leftrightarrow -a_{n-1} \cdot 2^{n-1} + \langle a[n-2:0]\rangle &< 0 \\ \Leftrightarrow \langle a[n-2:0]\rangle &< a_{n-1} \cdot 2^{n-1} \\ \Leftrightarrow a_{n-1} &\overset{(1)}{=} 1 \\ \text{da } 2^{n-1} > \langle a[n-2:0]\rangle &\geq 0 \quad (1)\end{aligned}$$

  Beweis der Hilfsbehauptung:

  - $n = 1$:

$$\langle a[-1:0]\rangle = 0 < 1 = 2^0$$

  - $n \rightsquigarrow n+1$:

$$\begin{aligned}\langle a[n-2:0]\rangle &= a_{n-2} \cdot 2^{n-2} + \langle a[n-3:0]\rangle \\ &\overset{IV}{<} a_{n-2} \cdot 2^{n-2} + 2^{n-2} \\ &\leq 2^{n-2} + 2^{n-2} = 2^{n-1}\end{aligned}$$

□

- $[a] \equiv \langle a[n-2:0]\rangle \mod 2^{n-1} \Leftrightarrow \exists k \in \mathbb{Z}: \; [a] - \langle a[n-2:0]\rangle = k \cdot 2^{n-1}$

$$\begin{aligned}[a] - \langle a[n-2:0]\rangle &= -a_{n-1} \cdot 2^{n-1} + \langle a[n-2:0]\rangle - \langle a[n-2:0]\rangle \\ &= -a_{n-1} \cdot 2^{n-1} \Rightarrow k = -a_{n-1}\end{aligned}$$

□

- $[a] \equiv \langle a\rangle \mod 2^n$

$$\begin{aligned}[a] - \langle a\rangle &= -a_{n-1} \cdot 2^{n-1} + \langle a[n-2:0]\rangle - \langle a[n-1:0]\rangle \\ &= -a_{n-1} \cdot 2^{n-1} + \langle a[n-2:0]\rangle - a_{n-1} \cdot 2^{n-1} - \langle a[n-2:0]\rangle \\ &= -2 \cdot a_{n-1} \cdot 2^{n-1} \\ &= -a_{n-1} \cdot 2^n \Rightarrow k = -a_{n-1}\end{aligned}$$

□

- $-[a] = [\overline{a}] + 1$

$$
\begin{aligned}
[\overline{a}] + 1 &= -\overline{a_{n-1}} \cdot 2^{n-1} + \left\langle \overline{a\,[n-2:0]} \right\rangle + 1 \\
&= -\overline{a_{n-1}} \cdot 2^{n-1} + (\sum_{i=0}^{n-2} \overline{a_i} \cdot 2^i) + 1 \\
&= -(1 - a_{n-1}) \cdot 2^{n-1} + (\sum_{i=0}^{n-2} (1 - a_i) \cdot 2^i) + 1 \\
&= -2^{n-1} + a_{n-1} \cdot 2^{n-1} + \sum_{i=0}^{n-2} 2^i - (\sum_{i=0}^{n-2} a_i \cdot 2^i) + 1 \\
&= -2^{n-1} + a_{n-1} \cdot 2^{n-1} + 2^{n-1} - 1 - (\sum_{i=0}^{n-2} a_i \cdot 2^i) + 1 \\
&= a_{n-1} \cdot 2^{n-1} - \sum_{i=0}^{n-2} a_i \cdot 2^i \\
&= -(-a_{n-1} \cdot 2^{n-1} + \langle a\,[n-2:0] \rangle) \\
&= -\,[a]
\end{aligned}
$$

$\square$

Aufgabe 4: cheap full adder

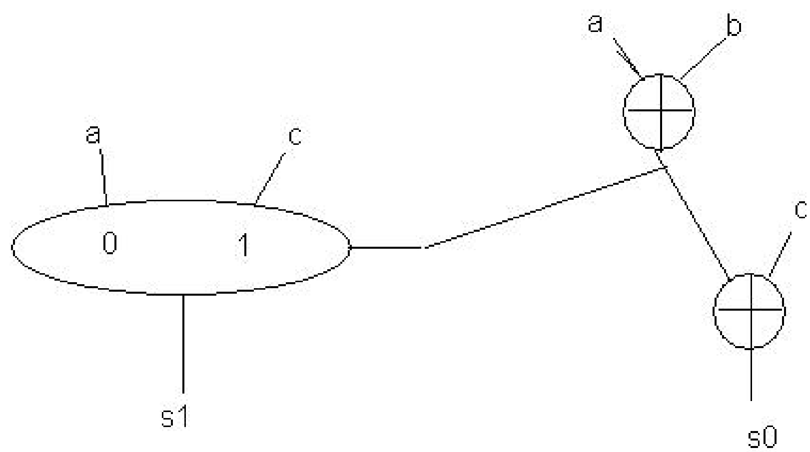

Abbildung 1: Full Adder

Aufgabe 5: arithmetic unit

$$\begin{aligned}
[a] &= -a_{n-1} \cdot 2^{n-1} + \langle a[n-2:0] \rangle \\
&= -2a_{n-1} \cdot 2^{n-1} + a_{n-1} \cdot 2^{n-1} + \langle a[n-2:0] \rangle \\
&= -a_{n-1} \cdot 2^{n} + \langle a \rangle
\end{aligned}$$

$$\begin{aligned}
[s] &= \begin{cases} [a] + [b] \mod 2^n & : sub = 0 \\ [a] - [b] \mod 2^n & : sub = 1 \end{cases} \\
\Leftrightarrow -s_{n-1} \cdot 2^n + \langle s \rangle &= \begin{cases} -a_{n-1} \cdot 2^n + \langle a \rangle + (-b_{n-1} \cdot 2^n + \langle b \rangle) \mod 2^n & : sub = 0 \\ -a_{n-1} \cdot 2^n + \langle a \rangle - (-b_{n-1} \cdot 2^n + \langle b \rangle) \mod 2^n & : sub = 1 \end{cases} \\
\Leftrightarrow \langle s \rangle &= \begin{cases} \langle a \rangle + \langle b \rangle \mod 2^n & : sub = 0 \\ \langle a \rangle - \langle b \rangle \mod 2^n & : sub = 1 \end{cases}
\end{aligned}$$