Bereits bewiesen

Seien $a, b, c \in \mathbb{N}$.

1. $a + 0 = a$ (n. Def. Addition)
2. $a + (b + 1) = (a + b) + 1$ (n. Def. Addition)
3. $0 + a = a$
   (a) $a = 0: \quad 0 + 0 \overset{(1)}{=} 0$
   (b) $a \to a + 1: \quad 0 + (a + 1) \overset{(2)}{=} (0 + a) + 1 \overset{IV}{=} a + 1$
4. $1 = 0 + 1$ (n. Def. 1)
5. $a + 1 = 1 + a$
   (a) $a = 0: \quad 0 + 1 \overset{(4)}{=} 1 \overset{(1)}{=} 1 + 0$
   (b) $a \to a + 1: \quad (a + 1) + 1 \overset{IV}{=} (1 + a) + 1 \overset{(2)}{=} 1 + (a + 1)$
6. $a + (b + c) = (a + b) + c$
   (a) $c = 0: \quad a + (b + 0) \overset{(1)}{=} a + b \overset{(1)}{=} (a + b) + 0$
   (b) $c \to c + 1: \quad a + (b + (c + 1)) \overset{(2)}{=} a + ((b + c) + 1) \overset{(2)}{=} (a + (b + c)) + 1 \overset{IV}{=} ((a + b) + c) + 1 \overset{(2)}{=} (a + b) + (c + 1)$
7. $a + b = b + a$
   (a) $b = 0: \quad a + 0 \overset{(1)}{=} a \overset{(3)}{=} 0 + a$
   (b) $b \to b + 1: \quad a + (b + 1) \overset{(2)}{=} (a + b) + 1 \overset{IV}{=} (b + a) + 1 \overset{(2)}{=} b + (a + 1) \overset{(5)}{=} b + (1 + a) \overset{(6)}{=} (b + 1) + a$
8. $a \cdot 0 = 0$ (n. Def. Multiplikation)
9. $a \cdot (b + 1) = a \cdot b + a$ (n. Def. Multiplikation)
10. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
    (a) $c = 0: \quad a \cdot (b + 0) \overset{(1)}{=} a \cdot b \overset{(1)}{=} (a \cdot b) + 0 \overset{(8)}{=} (a \cdot b) + (a \cdot 0)$
    (b) $c \to c + 1: \quad a \cdot (b + (c + 1)) \overset{(2)}{=} a \cdot ((b + c) + 1) \overset{(9)}{=} a \cdot (b + c) + a \overset{IV}{=} (a \cdot b + a \cdot c) + a \overset{(6)}{=} a \cdot b + (a \cdot c + a) \overset{(9)}{=} a \cdot b + a \cdot (c + 1)$
11. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
    (a) $c = 0: \quad a \cdot (b \cdot 0) \overset{(8)}{=} a \cdot 0 \overset{(8)}{=} 0 \overset{(8)}{=} (a \cdot b) \cdot 0$
    (b) $c \to c + 1: \quad a \cdot (b \cdot (c + 1)) \overset{(9)}{=} a \cdot (b \cdot c + b) \overset{(10)}{=} a \cdot (b \cdot c) + a \cdot b \overset{IV}{=} (a \cdot b) \cdot c + a \cdot b \overset{(9)}{=} (a \cdot b) \cdot (c + 1)$

12. $a^0 = 1$ (n. Def. Potenzierung)

13. $a^{b+1} = a^b \cdot a$ (n. Def. Potenzierung)

14. $a \cdot 1 = a$

    (a) $a \cdot 1 \overset{(4)}{=} a \cdot (0+1) \overset{(9)}{=} a \cdot 0 + a \overset{(8)}{=} 0 + a \overset{(3)}{=} a$

15. $0 \cdot a = 0$

    (a) $a = 0: \quad 0 \cdot 0 \overset{(8)}{=} 0$

    (b) $a \to a+1: \quad 0 \cdot (a+1) \overset{(9)}{=} 0 \cdot a + 0 \overset{IV}{=} 0 + 0 \overset{(1)}{=} 0$

16. $a \cdot b + b = (a+1) \cdot b$

    (a) $b = 0: \quad a \cdot 0 + 0 \overset{(1)}{=} a \cdot 0 \overset{(8)}{=} 0 \overset{(8)}{=} (a+1) \cdot 0$

    (b) $b \to b+1: \quad a \cdot (b+1) + (b+1) \overset{(9)}{=} (a \cdot b + a) + (b+1) \overset{(7)}{=} (a + a \cdot b) + (b+1) \overset{(2)}{=} ((a + a \cdot b) + b) + 1 \overset{(6)}{=} (a + (a \cdot b + b)) + 1 \overset{IV}{=} (a + (a+1) \cdot b) + 1 \overset{(7)}{=} ((a+1) \cdot b + a) + 1 \overset{(2)}{=} (a+1) \cdot b + (a+1) \overset{(9)}{=} (a+1) \cdot (b+1)$

17. $a \cdot b = b \cdot a$

    (a) $b = 0: \quad a \cdot 0 \overset{(8)}{=} 0 \overset{(15)}{=} 0 \cdot a$

    (b) $b \to b+1: \quad a \cdot (b+1) \overset{(9)}{=} a \cdot b + a \overset{IV}{=} b \cdot a + a \overset{(16)}{=} (b+1) \cdot a$

18. $a^{b+c} = a^b \cdot a^c$

    (a) $c = 0: \quad a^{b+0} \overset{(1)}{=} a^b \overset{(14)}{=} a^b \cdot 1 \overset{(12)}{=} a^b \cdot a^0$

    (b) $c \to c+1: \quad a^{b+(c+1)} \overset{(2)}{=} a^{(b+c)+1} \overset{(13)}{=} a^{b+c} \cdot a \overset{IV}{=} (a^b \cdot a^c) \cdot a \overset{(11)}{=} a^b \cdot (a^c \cdot a) \overset{(13)}{=} a^b \cdot a^{c+1}$

19. $(a \cdot b)^c = a^c \cdot b^c$

    (a) $c = 0: \quad (a \cdot b)^0 \overset{(12)}{=} 1 \overset{(14)}{=} 1 \cdot 1 \overset{(12)}{=} a^0 + b^0$

    (b) $c \to c+1: \quad (a \cdot b)^{c+1} \overset{(13)}{=} (a \cdot b)^c \cdot (a \cdot b) \overset{IV}{=} (a^c \cdot b^c) \cdot (a \cdot b) \overset{(11)}{=} ((a^c \cdot b^c) \cdot a) \cdot b \overset{(17)}{=} ((b^c \cdot a^c) \cdot a) \cdot b) \overset{(11)}{=} (b^c \cdot (a^c \cdot a)) \cdot b \overset{(13)}{=} (b^c \cdot a^{c+1}) \cdot b \overset{(17)}{=} (a^{c+1} \cdot b^c) \cdot b \overset{(11)}{=} a^{c+1} \cdot (b^c \cdot b) \overset{(13)}{=} a^{c+1} \cdot b^{c+1}$