Aufgabe 1

Seien $a, b, c \in \mathbb{N}$.

1. $a + 0 = a$ (n. Def. Addition)
2. $a + (b + 1) = (a + b) + 1$ (n. Def. Addition)
3. $0 + a = a$
   (a) $a = 0 : 0 + 0 \overset{(1)}{=} 0$
   (b) $a \Rightarrow a + 1 : 0 + (a + 1) \overset{(2)}{=} (0 + a) + 1 \overset{IV}{=} a + 1$
4. $1 = 0 + 1$ (n. Def. 1)
5. $a + 1 = 1 + a$
   (a) $a = 0 : 0 + 1 \overset{(4)}{=} 1 \overset{(1)}{=} 1 + 0$
   (b) $a \Rightarrow a + 1 : (a + 1) + 1 \overset{IV}{=} (1 + a) + 1 \overset{(2)}{=} 1 + (a + 1)$
6. $a + (b + c) = (a + b) + c$
   (a) $c = 0 : a + (b + 0) \overset{(1)}{=} a + b \overset{(1)}{=} (a + b) + 0$
   (b) $c \Rightarrow c + 1 : a + (b + (c + 1)) \overset{(2)}{=} a + ((b + c) + 1) \overset{(2)}{=} (a + (b + c)) + 1 \overset{IV}{=} ((a + b) + c) + 1 \overset{(2)}{=} (a + b) + (c + 1)$
7. $a + b = b + a$
   (a) $b = 0 : a + 0 \overset{(1)}{=} a \overset{(3)}{=} 0 + a$
   (b) $b \Rightarrow b + 1 : a + (b + 1) \overset{(2)}{=} (a + b) + 1 \overset{IV}{=} (b + a) + 1 \overset{6}{=} b + (a + 1) \overset{(5)}{=} b + (1 + a) \overset{(6)}{=} (b + 1) + a$

$\square$

Aufgabe 2

Seien $a, b, c \in \mathbb{N}$.

8. $a \cdot 0 = 0$ (n. Def. Multiplikation)
9. $a \cdot (b + 1) = a \cdot b + a$ (n. Def. Multiplikation)
10. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
    (a) $c = 0 : a \cdot (b + 0) \overset{(1)}{=} a \cdot b \overset{(1)}{=} (a \cdot b) + 0 \overset{(8)}{=} (a \cdot b) + (a \cdot 0)$
    (b) $c \Rightarrow c + 1 : a \cdot (b + (c + 1)) \overset{(2)}{=} a \cdot ((b + c) + 1) \overset{(9)}{=} a \cdot (b + c) + a \overset{IV}{=} (a \cdot b + a \cdot c) + a \overset{(6)}{=} a \cdot b + (a \cdot c + a) \overset{(9)}{=} a \cdot b + a \cdot (c + 1)$

$\square$

Aufgabe 3

Seien $a, b, c \in \mathbb{N}$.

11. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

    (a) $c = 0 : a \cdot (b \cdot 0) \overset{(8)}{=} a \cdot 0 \overset{(8)}{=} 0 \overset{(8)}{=} (a * b) * 0$

    (b) $c \Rightarrow c{+}1 : a \cdot (b \cdot (c{+}1)) \overset{(9)}{=} a \cdot (b \cdot c{+}b) \overset{(10)}{=} a \cdot (b \cdot c){+}a \cdot b \overset{IV}{=} (a \cdot b) \cdot c{+}a \cdot b \overset{(9)}{=} (a \cdot b) \cdot (c + 1)$

$\square$

Aufgabe 4

Seien $x_1, ..., x_n \in \{0, 1\}$
zu Zeigen: $f(x_1, ..., x_n) = (x_1 \wedge f(1, x_2, ..., x_n) \vee (\overline{x_1} \wedge f(0, x_2, ..., x_n))$

1. $x_1 = 0$

$$\begin{aligned} f(0, x_2, ..., x_n) &= \underbrace{(0 \wedge f(1, x_2, ..., x_n))}_{0} \vee \underbrace{(1 \wedge f(0, x_2, ..., x_n))}_{f(0,x_2,...,x_n)} \\ &= 0 \vee f(0, x_2, ..., x_n) \\ &= f(0, x_2, ..., x_n) \end{aligned}$$

2. $x_1 = 1$

$$\begin{aligned} f(1, x_2, ..., x_n) &= \underbrace{(1 \wedge f(1, x_2, ..., x_n))}_{f(1,x_2,...,x_n)} \vee \underbrace{(0 \wedge f(0, x_2, ..., x_n))}_{0} \\ &= f(1, x_2, ..., x_n) \vee 0 \\ &= f(1, x_2, ..., x_n) \end{aligned}$$

$\square$

Aufgabe 9

$$\begin{aligned} \bigwedge_{a \notin T(f)} c(a) = 0 &\Leftrightarrow \exists a \notin T(f) : c(a) = 0 \\ &\Leftrightarrow \exists a \notin T(f) : \bigvee_{i=1}^{n} X_i^{\overline{a_i}} = 0 \\ &\Leftrightarrow \exists a \notin T(f) : \forall i \in \{1..n\} : X_i^{\overline{a_i}} = 0 \\ &\Leftrightarrow \exists a \notin T(f) : \forall i \in \{1..n\} : X_i = a_i \\ &\Leftrightarrow \exists a \notin T(f) : X = a \\ &\Leftrightarrow X \notin T(f) \\ &\Leftrightarrow f(X) = 0 \end{aligned}$$

$\square$