SHAPE RECOGNITION, PRAIRIE FIRES, CONVEX DEFICIENCIES AND SKELETONS

L. CALABI AND W. E. HARTNETT, Parke Mathematical Laboratories

1. Introduction. Imagine an ideally homogeneous prairie, dry and ready to burn except for a very wet area A, with total absence of disturbing effects, as those of winds and slopes. Assume now that the grass at the edge of A is set afire, all at once, and observe how the fire develops. Since the shape of A is the only factor that we have not excluded, it and it alone will influence the spreading of the fire.

For instance, if A is a full circular region (a disc), each point outside A is eventually burned by fire moving radially away from A and continuing on, unchecked, in the same direction. If we now take for A two separate discs A_1, A_2, again the fire will eventually reach every point outside A, moving radially away from A. But now there are special points at which the radial spreading is checked: at these points the fire quenches itself. Assuming for simplicity that A_1 and A_2 have equal radii and centers O_1, O_2, the quench points are those equidistant from O_1 and O_2: in fact, at such points x the radial motion of the fire is stopped, since the rays O_1x and O_2x enter regions already burned.

For a last example, assume A to be the perimeter of a square field. The fire now will burn both the field and the prairie outside the fence; and it will quench itself at the points of the diagonals of the square, with the exception of the end-points.

Other experiments with such prairie fires (and we mean *Gedanken Experimenten*) should readily suggest that the presence of quench points is equivalent to the nonconvexity of the region A at whose edge the fire starts; further, that the set of quench points reflects some shape attributes of A.

Blum [2] had used the prairie fire as a model for a phase of the neurophysiological process of visual perception and suggested that a mathematical study of the correspondence between A and the set S of quench points, together with the function q expressing the time at which the fire reaches them, be undertaken.

In carrying out this study we have obtained results which clearly show that the pair (S, q) is a natural tool for the study of closed, nonconvex sets A in the Euclidean plane. Our work should thus be of interest to mathematicians even more perhaps than to firemen and neurophysiologists.

We present here, without proofs, the more important theorems, inviting the interested reader to obtain copies of the detailed treatment appearing in [**3**], [**4**], [**5**] and [**8**]. In Section 2 we introduce some of our basic tools and use a well-known theorem of Motzkin to identify closed convex sets as those for which there are no quench points. We define S and q in Section 3; and after an heuristic discussion of some examples, we characterize the family of sets A having a given pair (S, q). This theorem is an extension of Motzkin's theorem and identifies that property of A, called its convex deficiency, which is determined by the pair (S, q). The section terminates by discussing some situations in which

S and q uniquely determine A. In Section 4 we treat some intuitive properties of S and q and their interpretations in terms of A. In the paper we indicate some of the more obvious outstanding questions.

2. **Preliminaries.** We begin by recalling a number of standard notions about the Euclidean plane and some of its subsets. Throughout the rest of the paper A will always denote a nonvoid closed subset.

For each point x in the plane, the *distance* $r(x)$ *from* x *to* A is defined by $r(x) = \text{glb}\{d(x, y): y \in A\}$ where d is the usual Euclidean metric. Because A is closed, there always exists at least one point y of A "nearest" x such that $r(x) = d(x, y)$ and so the set $\pi x = \{y: y \in A, r(x) = d(x, y)\}$ is never void. Hence we have a function π defined on the plane with subsets of A as values; π is called the *projection onto* A, and πx the *projection of* x *onto* A. Notice that $\pi x = \{x\}$ if and only if $x \in A$.

Such set-valued functions have been studied by a number of authors (see [1], for example) and one may introduce the concepts of upper and lower semicontinuity as well as that of continuity. The projection π onto A is upper semicontinuous in the plane and is continuous on the set of points x such that πx is a singleton [7].

For each x we let $B(x)$ denote the closed ball of radius $r(x)$ and let $B^0(x)$ denote its interior. It is easy to see that $B^0(x)$ is nonvoid if and only if x belongs to the complement $-A$ of A. If $x \notin A$, we shall call $B(x)$ the *ball of support of* A *at* x. Observe that

$$\pi x = A \cap B(x) = \text{bd}(A) \cap B(x)$$

and $A \cap B^0(x) = \varnothing$, where $\text{bd}(A)$ denotes the boundary of A.

The function r has the property [6] that for x, x' in the plane

$$| r(x) - r(x') | \leqq d(x, x'),$$

that is, r is a Lipschitzian function with constant 1. If $x \notin A$, $y \in \pi x$, and $l = [y, x \rightarrow)$ is the closed ray through x with endpoint y, then it is easy to observe that $r(x) = r(x') + d(x, x')$ if $x' \in [y, x]$. This last equality is equivalent to the fact that $B(x') \subset B(x)$ with $\{y\} = A \cap B(x')$; that is, one ball of support is contained in the other and $y \in \pi x \cap \pi x'$. In terms of prairie fires this means that the fire burns along the ray l from y through x' and on to x. The natural question arises: is there a last point on the ray beyond which the fire will not spread? If so, then that point must be a quench point.

Before we try to answer the general question we look at a special case. Suppose A is a convex set, $x \notin A$, $y \in \pi x$, $l = [y, x \rightarrow)$ and x is the last point beyond which the fire on l will not spread. Pick a point x' beyond x on l, let $k = d(x', y)$ and construct a ball $B'(x')$ with center x' and radius k. Then y is on the boundary $B'(x')$ but the ball is not a ball of support. Hence there exists a point $y' \in A$ in the interior of $B'(x')$ such that $y' \in \pi x'$. But A is convex and so the closed interval $[y, y']$ belongs to A, which is impossible because $[y, y']$ contains points of $B^0(x)$ and $B(x)$ is a ball of support. Therefore, no last point exists.

Notice further that for each $x' \in l$, $\pi x' = \{y\}$ and so πx is a singleton for each x in the plane. Hence if $x' \in l$ there always exists other balls of support which contain $B(x')$ and so there are no balls of support which are maximal with respect to inclusion.

Our first theorem affirms that the converse implications also hold, as expected:

THEOREM 1. *The following statements are equivalent:*
(a) *A is a convex set;*
(b) *πx is a singleton for each x;*
(c) *There exist no balls of support that are maximal with respect to inclusion.*

As mentioned in the introduction, we could use a result of [10] proving the equivalence of (a) and (b). This equivalence can also be obtained as an immediate corollary of Theorem 2 below; and may be interpreted as the equivalence between the convexity of A and the absence of quench points. This interpretation will become more obvious after the definition of the set S in the next section.

3. The correspondence between A and (S, q). We are now ready to formalize the notion of quench point as the elements of what we shall call the skeleton of A. Recalling our earlier comments about the behaviour of the function r, we define the *skeleton S of A* as the set of those points $x \notin A$ for which

$$r(x) = r(x') + d(x, x') \qquad \text{if } x' \in [y, x],$$
$$r(x') < r(x) + d(x, x') \qquad \text{if } x' \in l - [y, x],$$

where $y \in \pi x$ and $l = [y, x \rightarrow)$.

The last inequality simply expresses the fact that for points x' beyond x on l there are points of A closer to x' than y. In the language of prairie fires, then, the burning reaches x' via another ray before it can reach it along l. Thus x is the last point of l burned by fire coming from y; that is, x is a quench point.

In terms of balls of support, it is easy to recognize that S is the set of those points $x \notin A$ at which $B(x)$ is maximal, that is not contained in any other ball of support. Both characterizations are useful in the development of the theory.

It might be worthwhile to observe that the first definition may be used to define the skeleton of any Lipschitzian function f with constant 1. We have only to replace r by f and to interpret the relation $y \in \pi x$ to mean that y belongs to the zero-set of f and to the boundary of the ball around x of radius $f(x)$. On the other hand, the second characterization of S may be used to define the skeleton of subsets A of metric spaces more general than the Euclidean plane.

Returning now to our subset A of the plane, observe that, by Theorem 1, S is empty if and only if A is convex. Thus if A is not convex, there are points in S. More specifically, if A is not convex there are points x at which πx contains at least two points (again by Theorem 1); and such points x belong to S because no ball of support can contain $B(x)$ unless it coincides with it. Those may not be all the points of S; however, as already shown [11], they are at least everywhere dense in S.

As already indicated in the introduction, besides its skeleton S we associate to A also the restriction q of r to S, called the *quench function* of A. The pair (S, q) will then be called the *skeletal pair* of A. For example, if A is convex then S is empty and so is q (we call empty the only function defined on the empty set); thus $(\varnothing, \varnothing)$ is the skeletal pair associated to each convex set A. Some examples with A nonconvex, chosen primarily to suggest and illustrate the results of this section on the correspondence $A \rightarrow (S, q)$ are given in Fig. 1. In the drawings, S is represented by dashed lines while A is drawn with heavy points, continuous lines and shaded areas. The value of q at some points x, x' of S is the length of the corresponding dotted and labelled segments; if $[y, x]$ is such a segment, then $y \in \pi x$ by the definition of q.

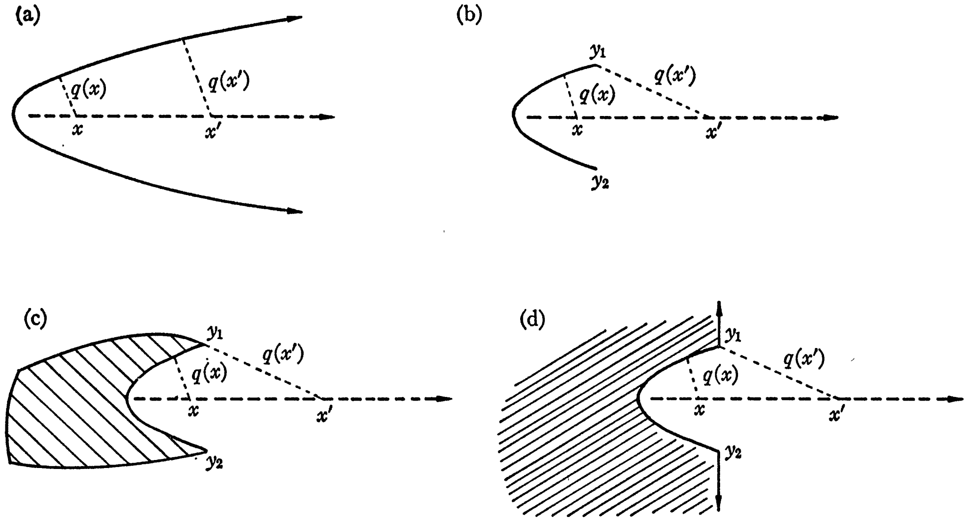

Fig. 1

The sets A illustrated in Fig. 1 are all constructed around the arc of the parabola y_1, y_2 of example (b), and all have the same skeleton S, the axis of parabola from the center of curvature at the vertex on. While examples (a) and (b) clearly have different quench functions q, the reader will readily realize that, as indicated, the last three examples do have the same quench function. Thus different sets A may have the same skeletal pairs, even if they are not convex.

Those examples, as well as the entire study so far, suggest that S and q could be determined by the manner in which A fails to be convex. In order to formulate this thought in a more precise manner, let us denote by C the closed convex hull of A and let us call the set $D = C \cap (-A) = C - A$ *the convex deficiency of* A. For the example (a) of Fig. 1, D is the region interior to the parabola; while for each of the other three examples D is part of the same set but does not extend beyond the open segment (y_1, y_2). Our examples then support the conjecture that the skeletal pair is more closely related to the convex deficiency D than to the set A itself. That this be indeed true is established in our central result, which may be considered as an extension of Motzkin's characterization of convex sets and of a theorem of Valentine [**15**, p. 96]:

THEOREM 2. *Two closed sets have the same skeletal pair if and only if they have the same convex deficiency.*

Thus, in suggestive language, not only the skeletal pair is determined by the "holes" and the "dents" of A, but also these, in turn, are determined by (S, q). In the more detailed studies [**4**] and [**5**] it has been shown how the holes of A correspond to connected components of S, and the dents of A to unbounded components of $S-C$.

Since it is possible to describe all the sets A having a given set D for their convex deficiency, we obtain a complete description of the correspondence $A \rightarrow (S, q)$ by way of the decomposition $A \rightarrow D \rightarrow (S, q)$. We do not describe these results here, preferring to emphasize the following corollaries of Theorem 2.

COROLLARY 3. *A closed set is uniquely determined by its convex hull and its skeletal pair.*

Our examples are good illustrations of this last result; in the light of it, example (b) suggests that there is but one set A having a given skeletal pair and satisfying $A^0 = \varnothing$. This is actually not true if A is contained in a line; but if $C^0 \neq 0$, then such a set A is the smallest having (S, q) as skeletal pair and is thus unique:

COROLLARY 4. *Assume $C^0 \neq \varnothing$ and $A^0 = \varnothing$. Then A is uniquely determined by its skeletal pair.*

This result is particularly important for the application of our theory to the visual recognition process: it establishes the adequacy of our tools for the characterization of "line figures," that is, of contours, the prime aim of Blum's investigation [**2**].

Corollary 4 may be obtained independently of Theorem 2 in the following way. We first show that the assumptions $C^0 \neq \varnothing$ and $A^0 = \varnothing$ imply $\overline{\pi S} = \mathrm{bd}(A) = A$, where $\overline{\pi S}$ denotes the closure of the union of πx for $x \in S$. We then establish that the restriction of π to S, and hence $\overline{\pi S}$, is explicitly determined by S and q: for $x \in S$, $\pi x = \{y: d(x, y) = q(x) \text{ and } d(x', y) \geqq q(x') \text{ for } x' \in S\}$.

In this context we might also quote the following more general statement:

COROLLARY 5. *Among the sets A having (S, q) as skeletal pair there is at most one for which* $\mathrm{bd}(A) = \overline{\pi S}$. *Thus, if there is one, it is uniquely determined by (S, q).*

We terminate this section by mentioning two open questions concerning the correspondence $A \rightarrow (S, q)$. The first is: what is the range of that mapping? In other words, if X is a subset of the plane and if f is a real valued function defined on it, under what conditions is (X, f) the skeletal pair of some set A? We know several rather stringent necessary but not sufficient conditions, some discussed below in Section 4; and we have obtained some sufficient but not necessary conditions. A satisfactory solution seems still far away.

The second and still wide-open problem may be introduced as follows.

If (S, q) can be used to recognize A (up to its convex hull C), "similar" sets A should have "neighboring" skeletal pairs. In more technical terms, let $\mathcal{A}$, $\mathcal{S}$, $\mathcal{S}^*$ be, respectively, the family of closed, nonconvex sets A of the plane, the family of their skeletal pairs (S, q) and the family of the triplets (S, q, C). Can we define topologies on these families such that the mapping $A \to (S, q, C)$ of $\mathcal{A}$ onto $\mathcal{S}^*$ be an homeomorphism and the induced mapping $A \to (S, q)$ of $\mathcal{A}$ onto $\mathcal{S}$ be continuous? Clearly the answer is yes: but can we do it in a natural manner? We feel that here "natural" is a very strict though still vague requirement, since it refers at once to mathematical and to visual criteria.

4. Two properties of S. In order to eliminate some of the pathology that can occur because of our weak assumptions on A, we shall restrict ourselves in this section to sets A such that $(\bar{S}-S) \subset A$. For this section only we say that such a set is a *good set.* All of the examples given so far have been good. An example of a nongood set A is a convergent sequence of points on a circle together with the limit point y. Each point on the open segment from the center x to y belongs then to $\bar{S}-S$ but clearly not to A.

In this example, the point x belongs to S but has no compact neighborhoods in S. Thus, if A is nongood, S may fail to be locally compact. It is easy to prove, on the other hand, that S is always "thin," at least insofar as $S^0 = \varnothing$. When A is good the situation is much nicer. Recall [16] that a *dendrite* is a compact, connected, and locally connected set which contains no simple closed curves; essentially, a dendrite is a nice union of simple arcs. We have then:

THEOREM 6. *Suppose A is good and $x \in S$. Then there exists a ball $B_0(x)$ with center x such that $S \cap B_0(x)$ is a dendrite.*

Thus S is not only locally compact and "thin," but even "graphlike." Actually we can say a bit more. Recall [16] that *the order $o(x)$ of x in S* is the smallest cardinal number, if it exists, such that each neighborhood of x in S contains another neighborhood whose boundary consists of $o(x)$ points. In a graph, $o(x)$ is defined for each x and is the number of branches issuing from x. Our theorem enables us to prove that the closure $\bar{S}$ of S is a graph (not necessarily bounded) when all of its points have finite order and when, say, $o(x) \neq 2$ for a finite number of points $x \in \bar{S}$. In all our examples, except for our nongood set, $\bar{S}$ is a graph. In the example of the square field, the vertices have order 1, the center has order 4, and all other points have order 2.

It is quite interesting to observe that the order $o(x)$ carries some information on what we could call the local shape of A. Denoting by $n(X)$ the number of connected components of a set X, one can show that, without any special assumption, $n(\pi x)$ never exceeds $o(x)$ (when defined) for x in S. Under our present assumptions we have more precisely:

THEOREM 7. *Suppose that A is good and $x \in S$. Then there exists a ball B around x such that $n(\pi x)$ equals $n((S \cap B)-\{x\})$. Moreover, if $o(x)$ is defined, then $n(\pi x)=o(x)$ whenever either of these numbers is finite.*

To interpret this theorem picturesquely, imagine that we stand at a point $x \in S$ whose order is finite. Then, in the ball B of the theorem, S consists of $o(x)$ branches leading into x. Because $n(\pi x)-o(x)$, the $n(\pi x)$ components of πx must occur "between" the branches of S (consider the example of the square with x the center). Hence, if we could look around us but only see points at exactly distance $q(x)$ away, then we would be able to see the $n(\pi x)$ connected subsets of $\text{bd}(A) \cap B(x)$ but no other points of A. Indeed, A must "bend away" from the circumference of $B(x)$ by having $o(x)$ "dents" or "cuts" as seen from x.

5. Concluding remarks. In presenting the material above we have considered only nonvoid closed sets in the plane with the usual metric. There are two obvious modifications possible. One may specialize and study sets with additional properties or one may generalize and investigate the situation in the setting of different spaces and/or different metrics.

The first path has been taken by Riley [**13**] and by Ting [**14**]. Riley limits himself to sets A which are graphs whose vertices have no points of accumulation and whose edges have a differentiable curvature with only finitely many changes of sign. Under these assumptions he shows in particular that $\bar{S}$ is itself a graph.

The work of Ting contains results that, in our context, suggest that the result of Riley may be strengthened. It appears that one may relax some of Riley's assumptions and still prove not only that $\bar{S}$ is a graph but that the edges are themselves differentiable.

The second path, already started by Motzkin and continued, for example, in [**9**] and [**12**] in connection with convex sets, seems also quite fruitful. The consideration of non-Euclidean metrics in the plane would moreover enhance the applicability of our studies to the neurophysiological process considered by Blum.

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ON MÜNTZ' THEOREM AND COMPLETELY MONOTONE FUNCTIONS

WILLIAM FELLER, Princeton University

1. Introduction. Weierstrass' theorem asserts that every continuous (real) function on $[0, 1]$ is the uniform limit of an appropriate sequence of polynomials. In a famous paper [7] Ch. Müntz considered the possibility of a similar approximation when not all powers of the coordinate variable are admitted, that is, an approximation by polynomials of the form

$$\sum_{k=1}^{n} c_k x^{\rho_k} \tag{1.1}$$

with prescribed exponents ρ_k. In this formulation it is not necessary that the ρ_k be integers, but the exponent 0 plays a special role being indispensable for approximations to the constant function. It is therefore preferable to consider only functions vanishing at the origin, and we denote the Banach space of such functions by $C_0[0, 1]$. As usual we put $\|f\| = \max|f(x)|$. Given a sequence of numbers ρ_k satisfying

$$0 < \rho_0 < \rho_1 < \cdots, \qquad \rho_k \to \infty, \tag{1.2}$$

Müntz' result may be stated as follows

THEOREM 1. *In order that the linear combinations of the form* (1.1) *be dense in* $C_0[0, 1]$ *it is necessary and sufficient that*

$$\sum \rho_k^{-1} = \infty. \tag{1.3}$$

In the case of nondenseness there exists a bounded linear functional annihilating all the powers x^{ρ_k}. But linear functionals on C_0 can be represented by signed measures on the half-open interval $(0, 1]$. Given such a (finite) measure μ we put

$$f(t) = \int_0^1 x^t \mu(dx). \tag{1.4}$$