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Lecture 25:
Unification of Denoising Methods

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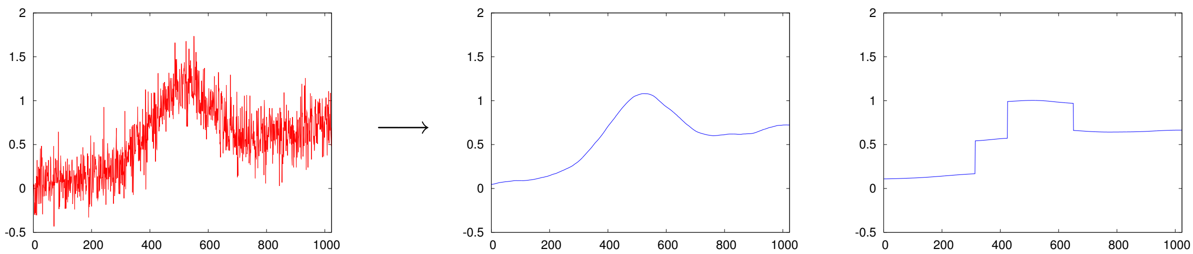
Motivation



- Many approaches for image simplication, smoothing and denoising exist, e.g.
  - local neighbourhood filters (mean, median)
  - nonlinear diffusion,
  - regularisation
- **Goals:**
  - embed various denoising methods into a unifying framework
  - formulation as minimisation of a discrete energy function

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From M-Estimators to Windowed Data Terms

Problem: Estimation of a Constant Signal

- given: measured noisy signal f_j: constant signal u with additive noise n_j

$$f_j = u + n_j \qquad (j \in J)$$

- goal: estimate signal u

M-Estimators

- minimise $E(u) = \sum_{j \in J} \Psi\left(|u - f_j|^2\right)$ with some increasing function Ψ

- no spatial information involved:
  global estimation that depends only on histogram of $\mathbf{f}$

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Examples of M-Estimators

- $\Psi(s^2) = s^2$ minimises the ℓ^2-distance $E(u) = \sum_{j \in J} |u - f_j|^2$.

  Yields the *arithmetic mean* which is well-suited for removing Gaussian noise.

- $\Psi(s^2) = s$ minimises the ℓ^1-distance $E(u) = \sum_{j \in J} |u - f_j|$.

  Yields the *median* which is good for removing impulse noise.

Problem

- M-smoothers yield a single estimator for the entire signal.

- only useful if a constant signal is to be estimated

From M-Estimators to Windowed Data Terms (3)

Local M-Smoothers (Chu et al. 1998)

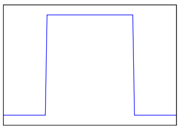
- introduce spatial windows in order to localise the estimation process
- find *local estimates* u_i from a local neighbourhood around position x_i:

$$u_i = \underset{y}{\text{argmin}} \sum_{j \in J} \Psi\left(|y - f_j|^2\right) \, w\left(|x_i - x_j|^2\right)$$

with a local neighbourhood or window $\mathcal{B}(i)$ generated by *spatial weight* w

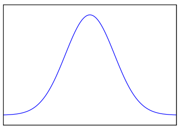
- Examples of window functions:

$$w(s^2) = \begin{cases} 1 & s^2 < \theta \\ 0 & \text{otherwise} \end{cases}$$



hard window
(Koenderink, van Doorn 1999)

$$w(s^2) = \exp\left(-\frac{s^2}{\theta^2}\right)$$



soft window
(Chu et al. 1998)

From M-Estimators to Windowed Data Terms (4)

- Since local M-smoothers estimate u_i via the optimisation

$$u_i = \underset{y}{\text{argmin}} \sum_{j \in J} \Psi\left(|y - f_j|^2\right) \, w\left(|x_i - x_j|^2\right),$$

they minimise the *windowed data term*

$$E_D(\mathbf{u}) = \sum_{i,j \in J} \Psi\left(|u_i - f_j|^2\right) \, w\left(|x_i - x_j|^2\right)$$

- How can this be implemented ?

A Fixed Point Iteration Scheme for Local M-Smoothers

- The minimiser of $E_D(\mathbf{u})$ necessary satisfies

$$0 \;=\; \frac{\partial E_D}{\partial u_i} \;=\; 2\sum_{j\in J} \Psi'\left(|u_i - f_j|^2\right)(u_i - f_j)\, w\left(|x_i - x_j|^2\right) \qquad \forall i \in J$$

- This is a nonlinear system of equations for the unknowns u_i with $i \in J$.
- Solving for the red u_i gives

$$u_i \;=\; \frac{\sum_{j\in J} \Psi'\left(|u_i - f_j|^2\right) w\left(|x_i - x_j|^2\right) f_j}{\sum_{j\in J} \Psi'\left(|u_i - f_j|^2\right) w\left(|x_i - x_j|^2\right)}$$

  under the assumptions $\Psi'(s^2) > 0$, $w(s^2) \geq 0$ for all $s \in \mathbb{R}$, and $w(0) > 0$.

- can be solved with fixed point iteration (*W-estimator*, Winkler et al. 1999):

$$u_i^0 := f_i, \qquad u_i^{k+1} := \frac{\sum_{j\in J} \Psi'\left(|u_i^k - f_j|^2\right) w\left(|x_i - x_j|^2\right) f_j}{\sum_{j\in J} \Psi'\left(|u_i^k - f_j|^2\right) w\left(|x_i - x_j|^2\right)}$$

From Bilateral Filters to Windowed Smoothness Terms

Bilateral Filtering

(Tomasi / Manduchi 1998)

- performs weighted averaging with spatial and tonal weights w and g:

$$u_i \;=\; \frac{\sum_{j\in J} g\left(|f_i - f_j|^2\right) w\left(|x_i - x_j|^2\right) f_j}{\sum_{j\in J} g\left(|f_i - f_j|^2\right) w\left(|x_i - x_j|^2\right)}$$

- typical weights: Gaussians
- results look similar to nonlinear diffusion filtering

From Bilateral Filters to Windowed Smoothness Terms (2)

Left: Original image, 325×356 pixels. **Right:** After bilateral filtering. Authors: Tomasi / Manduchi 1998.

From Bilateral Filters to Windowed Smoothness Terms (3)

Iterated Bilateral Filtering

- Using a tonal weight Ψ' and spatial weight w, iterative bilateral filtering becomes

$$u_i^0 := f_i$$

$$u_i^{k+1} := \frac{\sum_{j \in J} \Psi'\left(|u_i^k - u_j^k|^2\right) w\left(|x_i - x_j|^2\right) u_j^k}{\sum_{j \in J} \Psi'\left(|u_i^k - u_j^k|^2\right) w\left(|x_i - x_j|^2\right)}$$

- This has a structure that resembles the W-estimators

$$u_i^{k+1} := \frac{\sum_{j \in J} \Psi'\left(|u_i^k - f_j|^2\right) w\left(|x_i - x_j|^2\right) f_j}{\sum_{j \in J} \Psi'\left(|u_i^k - f_j|^2\right) w\left(|x_i - x_j|^2\right)}$$

- In a similar way as W-estimators minimise the windowed data term

$$E_D(\mathbf{u}) = \sum_{i,j \in J} \Psi\left(|u_i - f_j|^2\right)\, w\left(|x_i - x_j|^2\right)$$

  one can regard iterative bilateral filtering as minimisation of the *windowed smoothness term*

$$E_S(\mathbf{u}) = \sum_{i,j \in J} \Psi\left(|u_i - u_j|^2\right)\, w\left(|x_i - x_j|^2\right).$$

- This yields minimal energy for constant u.
  Thus, were are interested in the evolution, not the steady state.

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A Unifying Framework

Model

- windowed data and smoothness term as introduced before:

$$\begin{aligned} E_D(\mathbf{u}) &= \sum_{i,j \in J} \Psi_D\left(|u_i - f_j|^2\right)\, w_D\left(|x_i - x_j|^2\right) \\ E_S(\mathbf{u}) &= \sum_{i,j \in J} \Psi_S\left(|u_i - u_j|^2\right)\, w_S\left(|x_i - x_j|^2\right) \end{aligned}$$

- convex combination leads to unifying energy

$$E(\mathbf{u}) \;=\; c\, E_D(\mathbf{u}) \;+\; (1-c)\, E_S(\mathbf{u}) \qquad 0 \le c \le 1$$

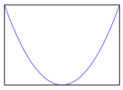
- allows to express a variety of image processing methods
- data term makes nonconstant minimisers possible for $c > 0$
- can be modified for interpolation by making c space-variant (cf. Lecture 15)

A Unifying Framework (2)

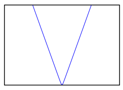
Penalisers

- Frequently used penalisers for the tonal weights in both the data and the smoothness term:

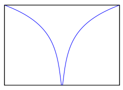
Whittaker–Tikhonov: $\Psi(s^2) = s^2$



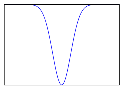
Regularised l^1: $\Psi(s^2) = 2\left(\sqrt{s^2+\varepsilon^2} - \varepsilon\right)$



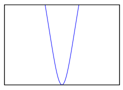
Perona-Malik: $\Psi(s^2) = \lambda^2 \log\left(1 + \frac{s^2}{\lambda^2}\right)$



Nonconvex: $\Psi(s^2) = \lambda^2\left(1 - \exp\left(-\frac{s^2}{\lambda^2}\right)\right)$



Truncated quadratic: $\Psi(s^2) = \min(s^2, \lambda^2)$



- The first two penalisers are convex and allow global minimisation.
  The others are nonconvex and may lead to local minima.

A Unifying Framework (3)

Important Special Case: Discrete Regularisation

- Using a quadratic local data term, and a smoothness term involving only the direct 4-neighbours $\mathcal{N}(i)$ gives

$$E(\mathbf{u}) = \sum_{i \in J}(u_i - f_i)^2 + \alpha \sum_{i \in J}\sum_{j \in \mathcal{N}(i)} \Psi\left(|u_i - u_j|^2\right)$$

- can be regarded as anisotropic variant of the isotropic nonquadratic regularisation

$$E(\mathbf{u}) = \sum_{i \in J}(u_i - f_i)^2 + \alpha \sum_{i \in J} \Psi\Big(\sum_{j \in \mathcal{N}(i)} |u_i - u_j|^2\Big)$$

  which approximates the continuous energy functional

$$E(u) = \int_\Omega \left((u-f)^2 + \alpha\, \Psi(|\nabla u|^2)\right) d\mathbf{x}.$$

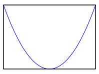
  and yields an implicit time discretisation of isotropic nonlinear diffusion filtering (cf. Lecture 10)

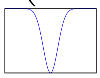
A Unifying Framework (4)

Application to Image Smoothing and Simplification

original image | quadratic Ψ_S | nonquadratic Ψ_S

noisy image | $\Psi_S(s^2) = s^2$ | $\Psi_S(s^2) = \lambda^2 \left(1 - \exp\left(-\frac{s^2}{\lambda^2}\right)\right)$





A Unifying Framework (5)

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Application of the Unifying Framework to Removal of Salt-and-Pepper Noise

$\longrightarrow$

- approximate median filtering:
  - hard windows w_D, w_S of size 3×3
  - (regularised) l^1 data term: $\Psi_D(s^2) = 2\left(\sqrt{s^2 + \varepsilon^2} - \varepsilon\right)$
  - Charbonnier smoothness term: $\Psi_S(s^2) = 2(\lambda^2\sqrt{1 + s^2/\lambda^2} - \lambda^2)$
  - data term dominates: $c = 0.95$

Summary

- Local M-smoothers minimise an energy with a windowed data term.
- A fixed point iteration of this energy yields W-estimators.
- Bilateral filtering performs spatial and tonal averaging.
- Iterative bilateral filtering can be regarded as a minimiser of an energy with a windowed smoothness term.
- Combining both energies gives a unified framework that includes
  - M-estimators
  - local M-smoothers (arithmetic mean, median)
  - W-estimators
  - bilateral filtering
  - discrete regularisation methods and diffusion filters

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  (parameter studies of the unified model)

Seminar in Winter Term 2007 / 2008: Modern Methods in Image Analysis

- offered by Dr. Andrés Bruhn and myself
- focusses on optic flow, stereo reconstruction, shape-from-shading, dynamic textures, image denoising, ...
- sign up by Thursday, July 10, 2008:

  http://www.mia.uni-saarland.de/Teaching/mmia08.shtml

- First come, first serve.
- First meeting:
  Friday, July 11, 2008, 2:15 pm, Building E1.1, Room 306.