Lecture 12:
Particle Image Velocimetry
Div-Curl-Regularisation, Incompressibility Constraint

Contents

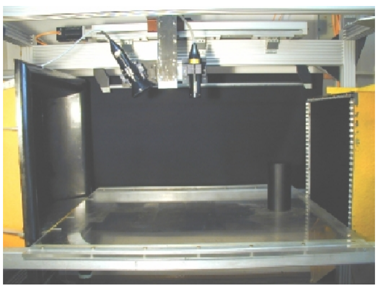
M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

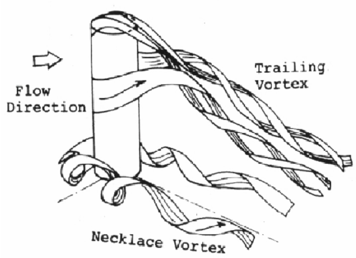
Particle Image Velocimetry

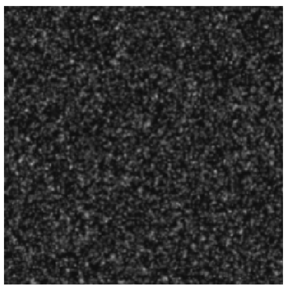
What is the PIV Problem?

Given: Two particle images that have been artificially created.
Wanted: Complex motion field with turbulent flows and nonstationary phenomena.

How Are PIV Images Obtained?





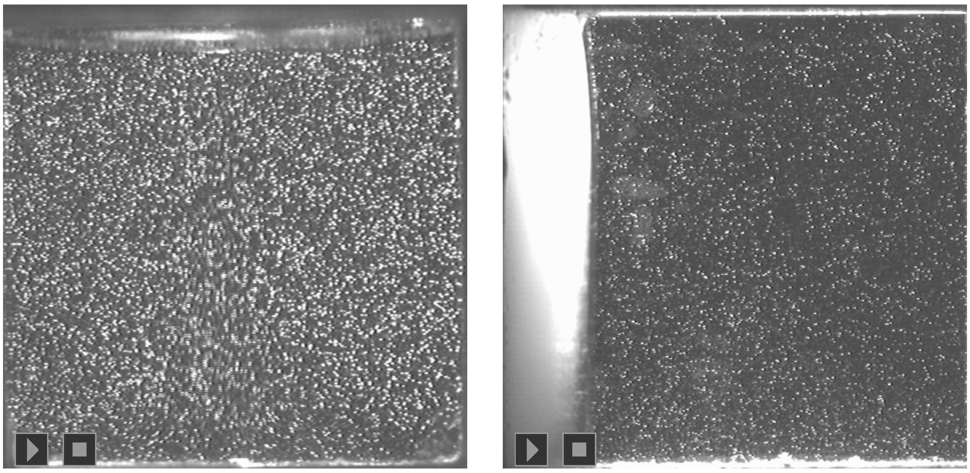


(a) Left: Experimental setting for studying the flow around a cylinder. **(b) Middle:** Schematic illustration of flow phenomena. **(c) Right:** Typical PIV image. *Authors:* P. Ruhnau (2006), T.Kawamura (1984).

Particle Image Velocimetry (2)

How Do Typical PIV Sequences Look Like?

- Video Example
  (*Quénot/Pakleza/Kowalewski 1998*)

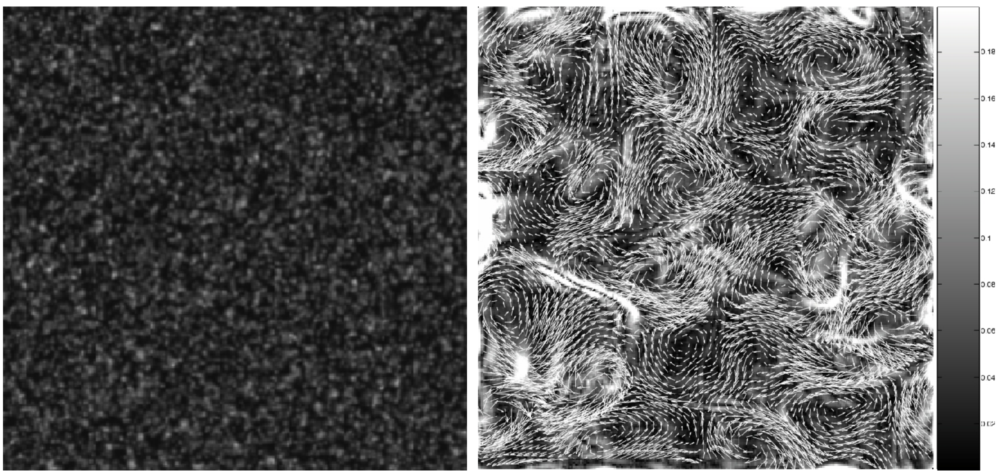


PIV Test Sequences. **(a) Left:** PIV Sequence 1. **(b) Right:** PIV Sequence 2.

Particle Image Velocimetry (3)

How Do Typical PIV Results Look Like?

- Estimated PIV Displacement Field
  (*Ruhnau/Stahl/Schnörr 2006*)



PIV Estimation. **(a) Left:** PIV Image. **(b) Right:** Corresponding flow field.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Fields of Application

- Aerodynamics (e.g. aeronautics, car industry)
- Fluid dynamics (e.g. process engineering, shipbuilding)
- Thermodynamics (e.g. machine/chemical engineering)
- Meteorology (e.g. climate analysis, weather forecast)

Challenges in PIV Data

- Small particles but large displacements
- Non-rigid motion (guided by physical laws)
- Complex motion patterns such as vortices
- Incompressibility constraints

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

First Order Div-Curl-Regularisation

- *Idea:* Penalise deviations from two kind of smoothness assumptions
- *Assumption 1:* Penalise **divergence** of flow field (no appearing and vanishing information → incompressibility constraint as in elastic regularisation)

$$\left(\operatorname{div}\begin{pmatrix}u\\v\end{pmatrix}\right)^2 = \left(\nabla\cdot\mathbf{u}\right)^2 = \left(u_x+v_y\right)^2.$$

- *Assumption 2:* Penalise **vorticity** of the flow field (no directional flow changes)

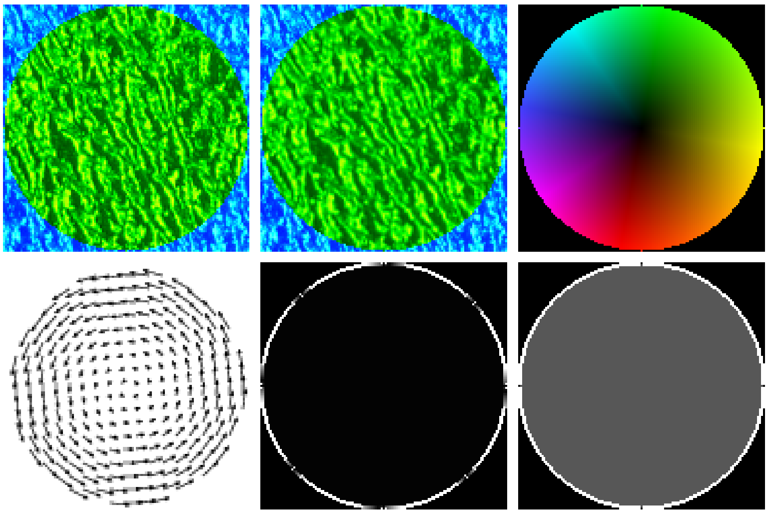
$$\left(\operatorname{curl}\begin{pmatrix}u\\v\end{pmatrix}\right)^2 = \left(\nabla\times\mathbf{u}\right)^2 = \left(v_x-u_y\right)^2.$$

- *Consequence*: This penalisation yields mainly **laminar flow fields** (parallel flow).
- *Remark:* Divergence and curl are so-called **first order differential invariants**.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Visualisation

- Divergence and Curl Component for the Slowly Rotating Disc Sequence



Visualisation of divergence and curl component of a flow field. **(a) Upper Left:** Frame 1 of Slowly Rotating Disc Sequence. **(b) Upper Center:** Frame 2. **(c) Upper Right:** Ground truth (colour plot). **(d) Lower Left:** Ground truth (vector plot). **(e) Lower Center:** Divergence. **(f) Lower Right:** Curl.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

First Order Div-Curl-Regularisation in Variational Approaches

- *Idea:* Combine divergence and curl penalisers in variational method
- *Example:* The energy functional for the Horn and Schunck method with **first order div-curl-regularisation** is given by
  (Suter 1994)

$$E(\mathbf{w}) = \int_\Omega \underbrace{(f_x\, u + f_y\, v + f_t)^2}_{\text{data term}} + \alpha \underbrace{\left((u_x + v_y)^2\right)}_{\text{divergence term}} + \beta \underbrace{\left((v_x - u_y)^2\right)}_{\text{curl term}} d\mathbf{x}\ .$$

- *Minimisation:* The corresponding Euler-Lagrange equations are given by the following **linear** equations

$$\begin{aligned} 0 &= J_{11}\, u + J_{12}\, v + J_{13} - \alpha\, (u_{xx} + v_{xy}) - \beta\, (u_{yy} - v_{xy})\ , \\ 0 &= J_{12}\, u + J_{23}\, v + J_{23} - \alpha\, (u_{xy} + v_{yy}) - \beta\, (v_{xx} - u_{xy}) \end{aligned}$$

  with associated **boundary conditions** that read

$$\mathbf{n}^\top \begin{pmatrix} \alpha(u_x + v_y) \\ \beta(u_y - v_x) \end{pmatrix} = 0 \quad \text{and} \quad \mathbf{n}^\top \begin{pmatrix} \beta(v_x - u_y) \\ \alpha(u_x + v_y) \end{pmatrix} = 0.$$

First Order Div-Curl-Regularisation (4)

First Order Div-Curl-Regularisation in Variational Approaches

- *Generic Form:* These Euler-Lagrange equations can be rewritten in the form

$$\begin{aligned} 0 &= J_{11}\,u + J_{12}\,v + J_{13} - \operatorname{div}\left(D_1\,\nabla u\right) - \operatorname{div}\left(D_2\,\nabla v\right)\,, \\ 0 &= J_{12}\,u + J_{22}\,v + J_{23} - \operatorname{div}\left(D_2\,\nabla u\right) - \operatorname{div}\left(D_3\,\nabla v\right)\,, \end{aligned}$$

  with boundary conditions as defined before.

- *Notation:* The **anisotropic** diffusion tensors D_1, D_2 and D_3 are given by

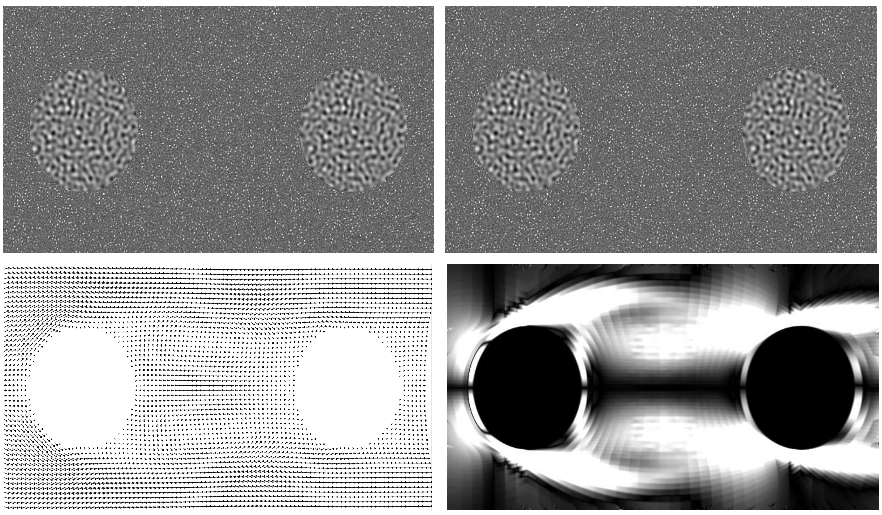
$$D_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\,, \quad D_2 = \begin{pmatrix} 0 & \frac{(\alpha-\beta)}{2} \\ \frac{(\alpha-\beta)}{2} & 0 \end{pmatrix}\,, \quad D_3 = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}\,.$$

- *Remark:* For the **special case** $\alpha = \beta$, the first order div-curl-regulariser collapses to the first order homogeneous regulariser of Horn and Schunck (**different b.c.**).

First Order Div-Curl-Regularisation (5)

First Order Div-Curl-Regularisation in Variational Approaches

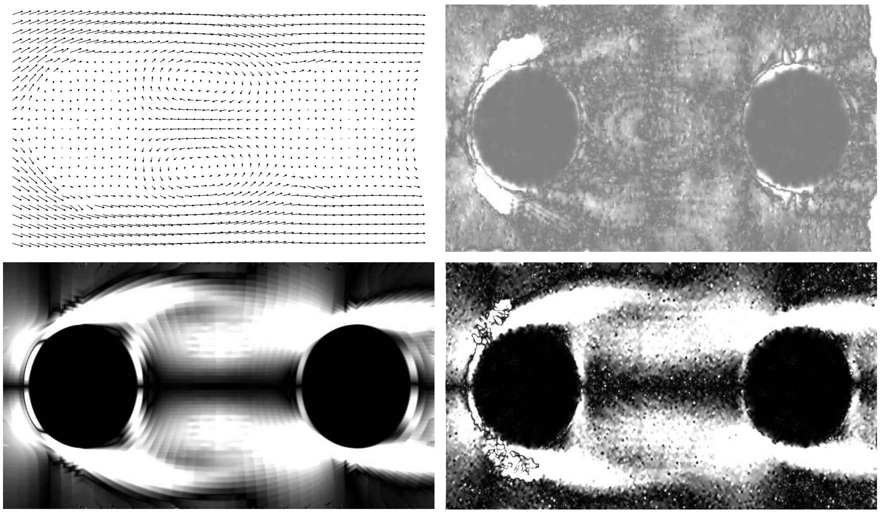
- Synthetic PIV Data with Two Cyclinders
  (*Quénot/Pakleza/Kowalewski 1998*)



Synthetic PIV data with two cylinders. **(a) Upper Left:** Frame 1. **(b) Upper Right:** Frame 4. **(c) Lower Left:** Ground truth. **(d) Outer Right:** Ground truth vorticity field (curl component of the flow).

First Order Div-Curl-Regularisation in Variational Approaches

- Results for Synthetic PIV Data with Two Cylinders
  (*Ruhnau/Kohlberger/Schnörr/Nobach 2005*)



(a) Upper Left: Estimated flow field. **(b) Upper Right:** Difference magnitude. **(c) Lower Left:** Ground truth vorticity field. **(d) Outer Right:** Estimated vorticity field.

Second Order Div-Curl-Regularisation

How Can We Improve the Behaviour w.r.t. Turbulent Flows?

- *Idea:* Assume divergence and vorticity to be smooth only (instead of being small)
- *Assumption 1:* Penalise deviations from **smoothness of divergence** via

$$\left|\nabla \operatorname{div} \begin{pmatrix} u \\ v \end{pmatrix}\right|^2 = |\nabla \, \nabla \cdot \mathbf{u}|^2 = (u_{xx} + v_{yx})^2 + (u_{xy} + v_{yy})^2 .$$

- *Assumption 2:* Penalise deviations from **smoothness of vorticity** via

$$\left|\nabla \operatorname{curl} \begin{pmatrix} u \\ v \end{pmatrix}\right|^2 = |\nabla \, \nabla \times \mathbf{u}|^2 = (v_{xx} - u_{yx})^2 + (v_{xy} - u_{yy})^2 .$$

- *Consequence*: In contrast to its first order variant, the second order div-curl-regularisation explicitly allows flow fields with (piecewise) **smooth** divergence and vorticity. In general, this yields much more realistic (turbulent) flow fields.

Second Order Div-Curl-Regularisation

- *Idea:* Combine second order divergence and curl penalisers in variational method
- *Example:* The energy functional for the Horn and Schunck method with **second order div-cur-regularisation** is given by
  (Suter 1994, Gupta/Prince 1996)

$$E(\mathbf{w}) = \int_\Omega \underbrace{(f_x u + f_y v + f_t)^2}_{\text{data term}} + \alpha \underbrace{\left((u_{xx}+v_{yx})^2 + (u_{xy}+v_{yy})^2\right)}_{\text{divergence term}} + \beta \underbrace{\left((u_{yx}-v_{xx})^2 + (u_{yy}-v_{xy})^2\right)}_{\text{curl term}} d\mathbf{x}\,,$$

where the weights α and β are positive weights.

- *Minimisation:* The corresponding **linear** Euler-Lagrange equations read

$$0 = J_{11}\,u + J_{12}\,v + J_{13} + \alpha\left((u_{xxxx}+v_{xxxy}) + (u_{xxyy}+v_{xyyy})\right) + \beta\left((u_{xxyy}-v_{xxxy}) + (u_{yyyy}-v_{xyyy})\right),$$
$$0 = J_{12}\,u + J_{22}\,v + J_{23} + \alpha\left((v_{xxyy}+u_{xxxy}) + (v_{yyyy}+u_{xyyy})\right) + \beta\left((v_{xxxx}-u_{xxxy}) + (v_{xxyy}-u_{xyyy})\right).$$

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Second Order Div-Curl-Regularisation

- *Boundary Conditions:* The associated **boundary conditions** that are given by

$$\mathbf{n}^\top \begin{pmatrix} \alpha(u_{xxx}+v_{xxy}+u_{xyy}+v_{yyy}) \\ \beta(u_{xxy}-v_{xxx}+u_{yyy}-v_{xyy}) \end{pmatrix} = 0\,,\quad \mathbf{n}^\top \begin{pmatrix} \alpha(u_{xx}+v_{xy}) \\ \alpha(u_{xy}+v_{yy}) \end{pmatrix} = 0\,,\quad \mathbf{n}^\top \begin{pmatrix} \beta(u_{xy}-v_{xx}) \\ \beta(u_{yy}-v_{xy}) \end{pmatrix} = 0\,,$$
$$\mathbf{n}^\top \begin{pmatrix} \beta(v_{xxx}-u_{xxy}+v_{xyy}-u_{yyy}) \\ \alpha(v_{xxy}+u_{xxx}+v_{yyy}+u_{xyy}) \end{pmatrix} = 0\,,\quad \mathbf{n}^\top \begin{pmatrix} \beta(v_{xx}-u_{xy}) \\ \beta(v_{xy}-u_{yy}) \end{pmatrix} = 0\,,\quad \mathbf{n}^\top \begin{pmatrix} \alpha(v_{xy}+u_{xx}) \\ \alpha(v_{yy}+u_{xy}) \end{pmatrix} = 0\,.$$

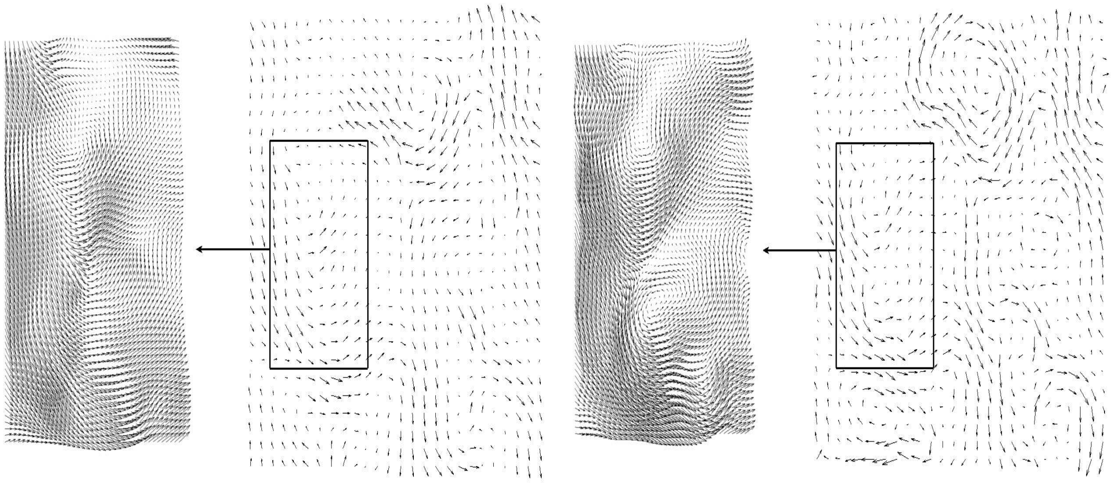
- *Remark:* For the case $\alpha = \beta$ we obtain the **linear** Euler-Lagrange equations

$$0 = J_{11}\,u + J_{12}\,v + J_{13} - \alpha\left(u_{xxxx} + 2u_{xxyy} + u_{yyyy}\right),$$
$$0 = J_{12}\,u + J_{22}\,v + J_{23} - \alpha\left(v_{xxxx} + 2v_{xxyy} + v_{yyyy}\right).$$

  These however, are exactly the Euler-Lagrange equations of the **second order homogeneous regulariser** (**different b.c.**) $\rightarrow$ curvature based regulariser.

Second Order Div-Curl-Regularisation in Variational Approaches

- Comparison of First vs. Second Order Div-Curl-Regularisation
(*Yuan/Ruhnau/Mémin/Schnörr 2005*)



(a) Left: First order div-curl-regulariser. **(b) Right:** Second order div-curl-regulariser.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Auxiliary Variables

How Can We Simplify Models with a Second Order Regulariser?

- *Idea:* Transform second order approach into first order approach
- *Example:* Defining the auxiliary variables

$$\xi = \operatorname{div}\begin{pmatrix} u \\ v \end{pmatrix} = (u_x + v_y) \quad \text{and} \quad \zeta = \operatorname{curl}\begin{pmatrix} u \\ v \end{pmatrix} = (v_x - u_y)$$

we can formulate the second order div-curl regularisation as
(Corpetti/Mémin/Pérez 2002)

$$E(\mathbf{w},\xi,\zeta) = \int_\Omega \underbrace{(f_x\,u + f_y\,v + f_t)^2}_{\text{data term}} + \alpha \underbrace{\left(\left(\xi - (u_x+v_y)\right)^2 + \lambda|\nabla\xi|^2\right)}_{\text{divergence term}} + \beta \underbrace{\left(\left(\zeta - (v_x-u_y)\right)^2 + \lambda|\nabla\zeta|^2\right)}_{\text{curl term}} d\mathbf{x}\,,$$

which only uses homogeneous first order regularisation for ξ and ζ.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Auxiliary Variables

- *Minimisation:* The corresponding Euler-Lagrange equations w.r.t. to u, v, ξ and ζ are given by the **linear** equations

$$\begin{aligned} 0 &= J_{11}\,u + J_{12}\,v + J_{13} - \alpha\,\big((u_{xx}+v_{xy}) - \xi_x\big) - \beta\,\big((u_{yy}-v_{xy}) - \zeta_y\big)\,, \\ 0 &= J_{12}\,u + J_{23}\,v + J_{23} - \alpha\,\big((u_{xy}+v_{yy}) - \xi_x\big) - \beta\,\big((v_{xx}-u_{xy}) + \zeta_y\big)\,, \\ 0 &= \big(\xi - (u_x+v_y)\big) - \lambda\Delta\xi\,, \\ 0 &= \big(\zeta - (v_x-u_y)\big) - \lambda\Delta\zeta\,, \end{aligned}$$

with associated **boundary conditions** for u and v that read

$$\mathbf{n}^\top \begin{pmatrix} \alpha\big((u_x+v_y)-\xi\big) \\ \beta\big((u_y-v_x)-\zeta\big) \end{pmatrix} = 0 \quad \text{and} \quad \mathbf{n}^\top \begin{pmatrix} \beta\big((v_x-u_y)+\zeta\big) \\ \alpha\big((u_x+v_y)-\xi\big) \end{pmatrix} = 0$$

and (reflecting) **Neumann boundary conditions** for ξ and ζ

$$\mathbf{n}^\top\nabla\xi = 0 \quad \text{and} \quad \mathbf{n}^\top\nabla\zeta = 0\,.$$

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Auxiliary Variables

- *Strategy:* Alternating solving of equations for u, v and ξ, ζ using two steps

- *Step 1:* Solve **one modified second order div-curl-regularisation problem** w.r.t. u^{k+1}, v^{k+1}

$$\begin{aligned} 0 &= J_{11}\,u^{k+1} + J_{12}\,v^{k+1} + J_{13} - \alpha\,\big((u^{k+1}_{xx}+v^{k+1}_{xy})\big) - \beta\,\big((u^{k+1}_{yy}-v^{k+1}_{xy})\big) + \big(\alpha\,\xi^k_x + \beta\,\zeta^k_y\big)\,, \\ 0 &= J_{12}\,u^{k+1} + J_{23}\,v^{k+1} + J_{23} - \alpha\,\big((u^{k+1}_{xy}+v^{k+1}_{yy})\big) - \beta\,\big((v^{k+1}_{xx}-u^{k+1}_{xy})\big) + \big(\alpha\,\xi^k_x - \beta\,\zeta^k_y\big)\,. \end{aligned}$$

The values for the auxiliary variables ξ and ζ are taken from the old iteration k.

- *Step 2:* Solve **two independent image regularisation problems** for ξ^{k+1}, ζ^{k+1}

$$\begin{aligned} 0 &= \big(\xi^{k+1} - (u^{k+1}_x + v^{k+1}_y)\big) - \lambda\Delta\xi^{k+1}\,, \\ 0 &= \big(\zeta^{k+1} - (u^{k+1}_y - v^{k+1}_x)\big) - \lambda\Delta\zeta^{k+1}\,. \end{aligned}$$

The values for the flow variables u and v are given by the previously computed results from the new iteration $k+1$, i.e. the values u^{k+1} and v^{k+1} are used.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Auxiliary Variables (4)

What is Image Regularisation?

- *Idea:* Create (piecewise) smooth version u of original image f
- *Assumption 1:* The smoothed version u shall be similar to the original image f. This can be expressed by the following **similarity term** with quadratic penaliser

$$(u-f)^2 \ .$$

- *Assumption 2:* The smoothed version u is assumed to be (piecewise) smooth. This can for instance be modelled using the quadratic **smoothness term**

$$|\nabla u|^2 \, .$$

- *Example:* Assuming similarity of u to the original image f and smoothness of the solution u one obtains the following energy functional

$$E(u) = \int_\Omega \underbrace{(u-f)^2}_{\text{similarity term}} + \alpha \underbrace{\left(|\nabla u|^2\right)}_{\text{smoothness term}} \, dx\, dy \ .$$

Auxiliary Variables (5)

Minimisation of Image Regularisation Functionals

- *Minimisation:* The corresponding Euler-Lagrange equation is the **linear** PDE

$$0 = \left(u - f\right) - \alpha \Delta u \ .$$

with (reflecting) Neumann boundary conditions $\mathbf{n}^\top \nabla u$ as known from optic flow.

- *Analogy:* One can easily verify that the following energy functional for ξ^{k+1}

$$E(\xi^{k+1}) = \int_\Omega \underbrace{\left(\xi^{k+1} - (u_x^{k+1} + v_y^{k+1})\right)^2}_{\text{similarity term}} + \lambda \underbrace{\left|\nabla \xi^{k+1}\right|^2}_{\text{smoothness term}} \, dx\, dy \ .$$

yields the Euler-Lagrange equation from the div-curl-regularisation with auxiliary variables

$$0 \;=\; \left(\xi^{k+1} - (u_x^{k+1} + v_y^{k+1})\right) - \lambda \Delta \xi^{k+1} \, .$$

The same can be shown for the Euler-Lagrange equation for the variable ζ^{k+1}.

Incompressible Navier Stokes Prior

- *Idea:* Integrate knowledge on the underlying physical process for fluid flows
- *Example:* Fluid flows are guided by the so-called **Navier Stokes equations**. They comprise one equation for the preservation of the momentum given by

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \mu \Delta \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mathbf{f}$$

  - $\mathbf{u}(\mathbf{x}) = (u, v)^\top$ velocity field (equals the displacement field for $\Delta t = 1$)
  - $\mathbf{f}(\mathbf{x})$ are external force fields that influence the flow (e.g. gravity)
  - $p(\mathbf{x})$ is the pressure field that steers the flow
  - $\rho(\mathbf{x})$ is the density field of the material(s)
  - λ, μ are viscosity constants (cf. Lamé constants in elastic regularisation)

  and two additional equations, one for the preservation of energy and one for the preservations of mass.

M I A

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Incompressible Navier Stokes Equations

- *Remark:* Assuming an **isotherm setting** (no heat sources or sinks) we can neglect the equation for the preservations of energy.
- *Assumption:* Assuming the **density ρ of particles not to change** along the path of motion we can rewrite the equation for the preservation of mass given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

  as the incompressibility constraint known from elastic regularisation

$$\rho(\nabla \cdot \mathbf{u}) = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{u} = 0\,.$$

- *Consequence:* We obtain the **incompressible Navier Stokes equations**

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\Delta \mathbf{u} + \frac{1}{\rho}\mathbf{f} \quad \rightarrow \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \tilde{p} + \nu\, \Delta \mathbf{u} + \tilde{\mathbf{f}}\,.$$

Incompressible Vorticity Transport Equation

- *Reformulation:* Assuming only conservative body forces (i.e. $\nabla \times \tilde{\mathbf{f}} = 0$) and applying the curl operator $\nabla\times$ (vorticity) to the resulting equation yields

$$\nabla \times \frac{\partial \mathbf{u}}{\partial t} + \nabla \times (\mathbf{u} \cdot \nabla)\mathbf{u} = -\underbrace{\nabla \times \nabla \tilde{p}}_{=\ 0} + \nabla \times \nu\, \Delta \mathbf{u} + \underbrace{\nabla \times \tilde{\mathbf{f}}}_{=\ 0} .$$

Taking into account the incompressibility constraint $\nabla \cdot \mathbf{u} = 0$ one obtains

$$\frac{\partial(\nabla \times \mathbf{u})}{\partial t} + \mathbf{u} \cdot \nabla(\nabla \times \mathbf{u}) = \nu\, \Delta(\nabla \times \mathbf{u}) .$$

Introducing a separate variable for the vorticity given by $w = \nabla \times \mathbf{u}$ yields finally the **incompressible vorticity transport equation**

$$\frac{\partial w}{\partial t} = \nu\, \Delta w - \mathbf{u} \cdot \nabla w .$$

- *Remark:* Knowing an initial vorticity $w_0 = w(\mathbf{x}, 0)$, we can compute a **predictor** $w_t = w(\mathbf{x}, t)$ at time t via the incompressible vorticity transport equation.

Variational Method with Incompressible Vorticity Transport Prior

- *Idea:* Use predictor $w_t = w(\mathbf{x}, t)$ as prior for the curl regulariser
- *Example:* Integrating the predictor $w_t = w(\mathbf{x}, t)$ as prior into a Horn and Schunck method with second order div-curl-regulariser yields for instance *(Ruhnau/Stahl/Schnörr 2006, Ruhnau/Stahl/Schnörr 2007)*

$$E(\mathbf{w}, w, d) = \int_\Omega \underbrace{\left(f_x\, u + f_y\, v + f_t\right)^2}_{\text{data term}} + \lambda\, \underbrace{\left((w - w_t)^2\right)}_{\text{vorticity prior}} + \kappa\, \underbrace{\left(|\nabla w|^2\right)}_{\text{curl term}} + \mu\, \underbrace{\left(|d|^2\right)}_{\text{div term}}\, d\mathbf{x} .$$

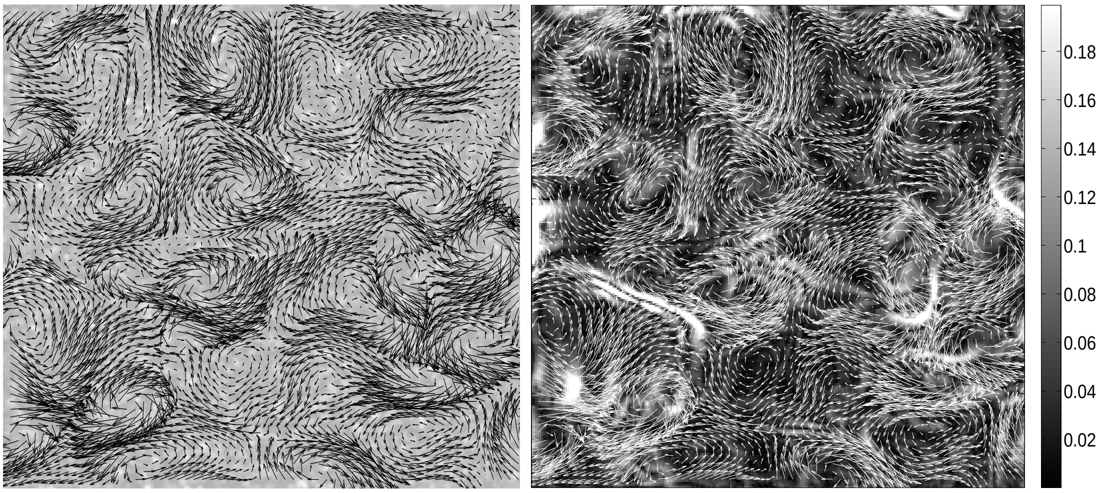
which has to be minimised subject to the conditions $d = \nabla \cdot \mathbf{u}$ and $w = \nabla \times \mathbf{u}$. The incompressibility constraint requires d to be small (not smooth).

- *Remark:* One may introduce two additional terms (as shown in previous section) that actually couple the parameters u and v with the auxiliary variables w and d.
- *Strategy:* Knowing the vorticity from the **previous frame** we can compute a prior for the **current frame** via the incompressible vorticity transport equation. Thus whole PIV sequences can be computed efficiently online, i.e. frame by frame.

Incompressible Navier Stokes Prior (5)

Incompressible Navier Stokes Prior in Variational Approaches

- Results for a Synthetic PIV Image Sequence
  (*Ruhnau/Stahl/Schnörr 2007*)

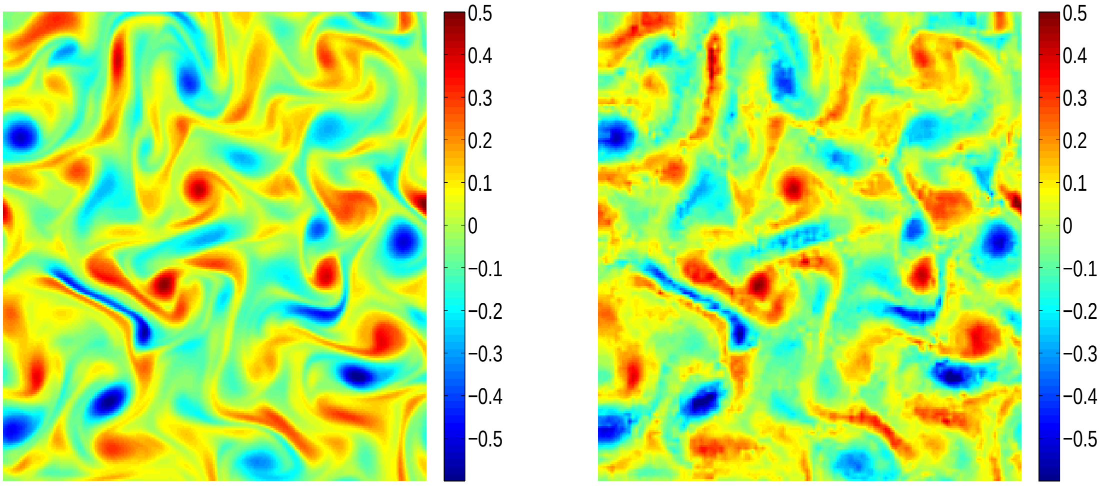


(a) Left: Ground truth flow. **(b) Right:** Estimated fluid flow with physical prior.

Incompressible Navier Stokes Prior (6)

Incompressible Navier Stokes Prior in Variational Approaches

- Results for a Synthetic PIV Image Sequence
  (*Ruhnau/Stahl/Schnörr 2007*)

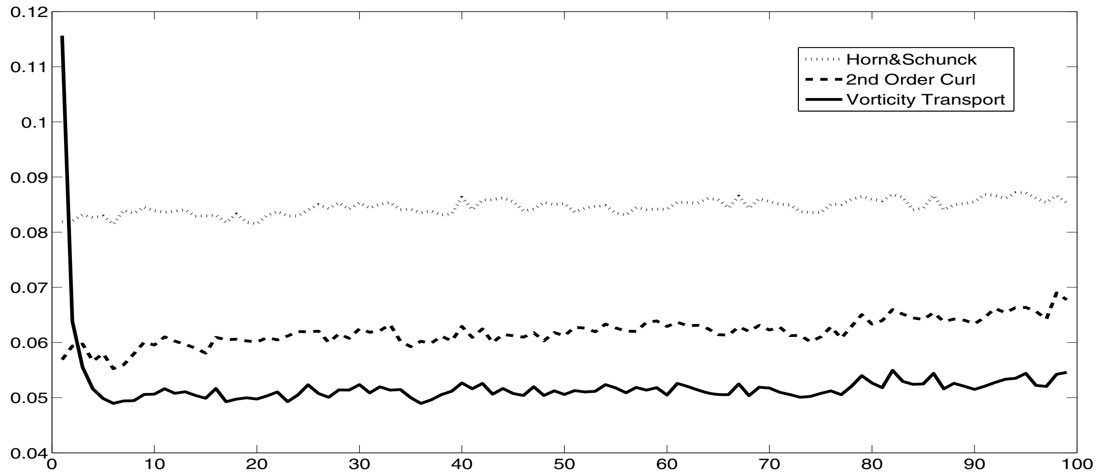


(a) Left: Ground truth vorticity. **(b) Right:** Estimated fluid flow with physical prior.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Incompressible Navier Stokes Prior in Variational Approaches

- Comparison of the Root Mean Squared Error for Different PIV Techniques (*Ruhnau/Stahl/Schnörr 2006*)



M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Summary

- First order div-curl-regularisers give reasonable results for mainly laminar flows (hardly div and curl components).
- Second order div-curl-regularisers can handle turbulent flows much better (by assuming only smoothness on div and curl components).
- Auxiliary variables allow to transform a second order regularisation problem into several problems with first order regularisation (at the expense of an alternating minimisation).
- Considering physical priors based on the Navier Stokes equations allows to improve the results even further (e.g. by predicting the vorticity).

M I A
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15 16
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