Lecture 9:

Stereo Reconstruction II
Estimation of the Fundamental Matrix

Contents

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Basic Idea

Which Constraints Are Available for Estimating the Fundamental Matrix?

- *Idea:* Let us consider the epipolar constraint given by the equation

$$\widetilde{\mathbf{m}}_2^\top \mathcal{F} \, \widetilde{\mathbf{m}}_1 = \begin{pmatrix} u_2 \\ v_2 \\ 1 \end{pmatrix}^\top \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ 1 \end{pmatrix} = 0 \,.$$

  Defining the following two vectors for correspondences and matrix entries

$$\begin{aligned} \mathbf{s} &= (u_1u_2, v_1u_2, u_2, u_1v_2, v_1v_2, v_2, u_1, v_1, 1)^\top \\ \mathbf{f} &= (f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33})^\top \end{aligned}$$

  we can write the epipolar constraint in terms of a **scalar product**

$$\widetilde{\mathbf{m}}_2^\top \mathcal{F} \, \widetilde{\mathbf{m}}_1 = \mathbf{s}^\top \mathbf{f} = 0 \,.$$

Basic Idea (2)

Two Types of Available Constraints

- *Correspondence Constraints:* For each known correspondence $(u_1^i, v_1^i) \leftrightarrow (u_2^i, v_2^i)$ we obtain one linear constraint

$$\mathbf{s}_i^\top \mathbf{f} = 0.$$

- *Rank-2 Constraint:* Additionally, we know that $\mathcal{F}$ has rank 2. Thus we obtain the following nonlinear (cubic) constraint

$$\begin{aligned} \det(\mathcal{F}) \quad = \quad & f_{11}f_{22}f_{33} + f_{12}f_{23}f_{31} + f_{13}f_{21}f_{32} \\ & -f_{31}f_{22}f_{13} - f_{32}f_{23}f_{11} - f_{33}f_{21}f_{12} = 0\ . \end{aligned}$$

- *Consequence:* We have two possibilities
  - use 8 correspondence constraints and enforce rank-2 constraint a posteriori ($\rightarrow$ **linear problem**)
  - use 7 correspondence constraints and the rank-2 constraint ($\rightarrow$ **nonlinear problem**, not discussed in this lecture)

Basic Idea (3)

Many Degrees of Freedom

- *Correspondences:* How to obtain the required point correspondences?
  - no search space restriction ($\rightarrow$ optic flow problem)
  - feature-based techniques (sparse but accurate results)
  - variational methods (dense/robust correspondences)
- *Extensibility:* Can we use more than 8 correspondences to improve the result?
  - least squares fit, total least square fit
  - random sampling techniques (RANSAC)
- *Penaliser:* How to penalise deviations from the correspondence constraints?
  - quadratically (nonrobust, linear problem)
  - sub-quadratically (robust, nonlinear problem)
  - geometrically (distance to epipolar lines, nonlinear problem)

Total Least Squares

- *Idea:* Use $N \geq 8$ correspondence constraints and solve the resulting linear systems of equations given by

$$S\,\mathbf{f} = 0\ .$$

The lines of the $N \times 9$ matrix S are formed by the different constraints

$$S = \begin{pmatrix} \mathbf{s}_1^\top \\ \vdots \\ \mathbf{s}_N^\top \end{pmatrix}\ .$$

- *Problems:* There are three main problems
  - trivial solution $\mathbf{f} = 0$ always works (homogeneous system of equations)
  - no solution possible for $N > 8$ constraints (system is over-determined)
  - infinitely many solutions for less than 8 linearly independent constraints (can be avoided in advance by picking a suitable set of correspondences)

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Total Least Squares Fit (TLS)

- *Idea:* Perform a **total least squares fit** based on the quadratic penalizer function *(Longuet-Higgins 1981)*

$$E(\mathbf{f}) = \sum_{i=1}^{N} (\mathbf{s}_i^\top \mathbf{f})^2$$

- *Modification:* Avoid the trivial solution by using the **side constraint** $|\mathbf{f}| = 1$ as Lagrangian multiplier $C(\mathbf{f}) = (1 - \mathbf{f}^\top \mathbf{f})$:

$$E^*(\mathbf{f}) = \underbrace{\sum_{i=1}^{N} (\mathbf{s}_i^\top \mathbf{f})^2 + \lambda(1 - \mathbf{f}^\top \mathbf{f})}_{E(\mathbf{f}) + \lambda C(\mathbf{f})} = \mathbf{f}^\top \left(\sum_{i=1}^{N} \mathbf{s}_i \mathbf{s}_i^\top \right) \mathbf{f} + \lambda(1 - \mathbf{f}^\top \mathbf{f}) = \mathbf{f}^\top S^\top S\,\mathbf{f} + \lambda(1 - \mathbf{f}^\top \mathbf{f})$$

- *Attention:* In order to find the minimiser of $E(\mathbf{f})$ subject to the constraint $C(\mathbf{f})$, we have to determine the saddle point and not the minimiser of $E^*(\mathbf{f})$.

Total Least Squares (3)

Solution as Eigenvalue Problem

- *Solution:* Differentiating with respect to $\mathbf{f}$, λ yields the **eigenvalue problem**

$$(S^\top S - \lambda I)\, \mathbf{f} = 0\,, \quad |\mathbf{f}| = 1$$

as necessary condition. The solution that actually minimises $E(\mathbf{f})$ is given by the eigenvector $\mathbf{e}$ to the smallest eigenvalue λ of the symmetric 9×9 matrix

$$S^\top S = \sum_{i=1}^{N} \mathbf{s}_i \mathbf{s}_i^\top \,.$$

- *Remark:* In the case of $N = 8$ the smallest eigenvalue is given by $\lambda = 0$.

Unfortunately, this technique is not robust w.r.t. outliers and has no geometrical interpretation. Let us first tackle the sensitivity to outliers.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

M-Estimators (1)

M-Estimators

- *Frequent Observation:* Often several outliers are among the correspondences. How can we handle this outliers appropriately during the estimation?
- *Idea:* Reduce influence of outliers by sub-quadratic penalisation

Left: Left image of the Herve stereo pair. **Right:** Right image with computed SIFT correspondences.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

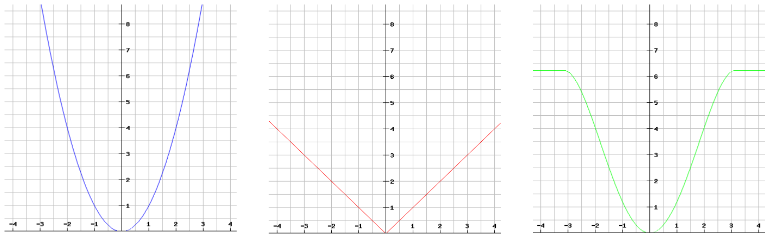
M-Estimators (2)

M-Estimator for the Total Least Squares Fit (TLS)

- *TLS Example:* Replace quadratic penaliser by linear one (cf. robust data terms)

$$E(\mathbf{f})^* = \sum_{i=1}^{N} (\mathbf{s}_i^\top \mathbf{f})^2 + \lambda(1 - \mathbf{f}^\top \mathbf{f}) \quad \Rightarrow \quad E(\mathbf{f})^* = \sum_{i=1}^{N} \Psi\Big((\mathbf{s}_i^\top \mathbf{f})^2\Big) + \lambda(1 - \mathbf{f}^\top \mathbf{f})$$

where $\Psi(s^2) = \sqrt{s^2 + \epsilon^2}$ is the regularised L_1 norm with small $\epsilon > 0$.



- *Properties:* In general $\Psi(s^2)$ should be **positive** and **sub-quadratic**. If a unique solution is desired $\Psi(s^2)$ should also be **strictly convex**. Nonconvex functions, however, are often more robust w.r.t. outliers (but more difficult to minimise).

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

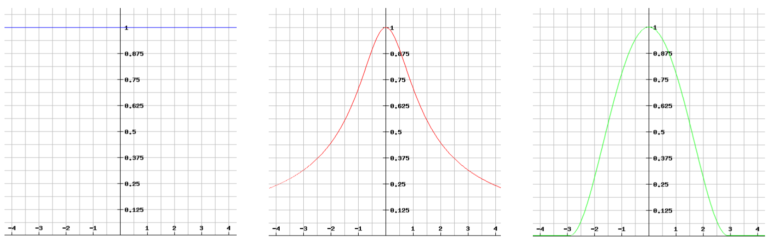
M-Estimators (3)

M-Estimator for the Total Least Squares Fit (TLS)

- *Solution:* Differentiating with respect to $\mathbf{f}$, λ yields the **nonlinear system**

$$\left(\sum_{i=1}^{N} \Psi'\Big((\mathbf{s}_i^\top \mathbf{f})^2\Big)\ \mathbf{s}_i\, \mathbf{s}_i^\top - \lambda I\right) \mathbf{f} = \left(S^\top W S - \lambda I\right) \mathbf{f} = 0, \quad |\mathbf{f}| = 1$$

where W is a diagonal weighting matrix with weights $w_{ii} = \Psi'\Big((\mathbf{s}_i^\top \mathbf{f})^2\Big)$.



We solve this system in a **lagged nonlinearity** fashion, since for fixed weights, we obtain a simple eigenvalue problem with respect to the symmetric 9×9 matrix

$$S^\top W S = \sum_{i=1}^{N} \Psi'\Big((\mathbf{s}_i^\top \mathbf{f})^2\Big)\ \mathbf{s}_i\, \mathbf{s}_i^\top \ .$$

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Geometric Approaches

- *Problem:* The expression $\tilde{\mathbf{m}}_2^{i\top} \mathcal{F} \tilde{\mathbf{m}}_1^i$ does not have a geometrical meaning
- *Idea:* Consider the minimisation of the distance of each point $\tilde{\mathbf{m}}_2^i$ to its epipolar line $\mathbf{l}_2^i$. Thus instead of the original linear distance measure given by

$$d_{\mathrm{L}}(\tilde{\mathbf{m}}_2^i, \mathcal{F}\,\tilde{\mathbf{m}}_1^i) = \tilde{\mathbf{m}}_2^{i\top} \mathcal{F}\,\tilde{\mathbf{m}}_1^i = \tilde{\mathbf{m}}_2^{i\top}\,\mathbf{l}_2^i$$

we obtain a nonlinear distance measure that can be interpreted in a geometrical sense. This nonlinear distance measure is given by the expression
(Luong/Faugeras 1996)

$$d_{\mathrm{NL}}(\tilde{\mathbf{m}}_2^i, \mathcal{F}\,\tilde{\mathbf{m}}_1^i) = \frac{1}{\sqrt{(\mathcal{F}\,\tilde{\mathbf{m}}_1^i)_1^2 + (\mathcal{F}\,\tilde{\mathbf{m}}_1^i)_2^2}}\;\tilde{\mathbf{m}}_2^{i\top} \mathcal{F}\,\tilde{\mathbf{m}}_1^i = \frac{\tilde{\mathbf{m}}_2^{i\top}\,\mathbf{l}_2^i}{\sqrt{(\mathbf{l}_2^i)_1^2 + (\mathbf{l}_2^i)_2^2}}$$

with $(\mathbf{l}_2^i)_1$ and $(\mathbf{l}_2^i)_2$ being the two first entries of the epipolar line $\mathbf{l}_2^i$ defined as

$$\mathbf{l}_2^i = \begin{pmatrix} (\mathbf{l}_2^i)_1 \\ (\mathbf{l}_2^i)_2 \\ (\mathbf{l}_2^i)_3 \end{pmatrix} = \begin{pmatrix} (\mathcal{F}\,\tilde{\mathbf{m}}_1^i)_1 \\ (\mathcal{F}\,\tilde{\mathbf{m}}_1^i)_2 \\ (\mathcal{F}\,\tilde{\mathbf{m}}_1^i)_3 \end{pmatrix} = \mathcal{F}\,\tilde{\mathbf{m}}_1^i\ .$$

Variants of Geometric Approaches

- *Simple Approach:* This approach minimises the distance of points in the right frame to their epipolar lines by minimising

$$E(\mathcal{F}) = \sum_{i=1}^{N} d_{\mathrm{NL}}^2(\tilde{\mathbf{m}}_2^i, \mathcal{F}\,\tilde{\mathbf{m}}_1^i) = \sum_{i=1}^{N} \frac{1}{(\mathcal{F}\,\tilde{\mathbf{m}}_1^i)_1^2 + (\mathcal{F}\,\tilde{\mathbf{m}}_1^i)_2^2}\left(\tilde{\mathbf{m}}_2^{i\top} \mathcal{F}\,\tilde{\mathbf{m}}_1^i\right)^2 .$$

- *Symmetrical Approach:* Here, the distances of the points to their epipolar lines in both frames are minimised simultaneously

$$E(\mathcal{F}) = \sum_{i=1}^{N}\left(d_{\mathrm{NL}}^2(\tilde{\mathbf{m}}_2^i, \mathcal{F}\,\tilde{\mathbf{m}}_1^i) + d_{\mathrm{NL}}^2(\tilde{\mathbf{m}}_1^i, \mathcal{F}^\top\,\tilde{\mathbf{m}}_2^i)\right) = \sum_{i=1}^{N}\left(\frac{1}{(\mathcal{F}\,\tilde{\mathbf{m}}_1^i)_1^2 + (\mathcal{F}\,\tilde{\mathbf{m}}_1^i)_2^2} + \frac{1}{(\mathcal{F}^\top\,\tilde{\mathbf{m}}_2^i)_1^2 + (\mathcal{F}^\top\,\tilde{\mathbf{m}}_2^i)_2^2}\right)\left(\tilde{\mathbf{m}}_2^{i\top} \mathcal{F}\,\tilde{\mathbf{m}}_1^i\right)^2 .$$

- *Gradient Approach:* Here, the fundamental matrix $\mathcal{F}$ is considered as a surface in a 4-dimensional space that must be fitted to the points $(u_1^i, v_1^i, u_2^i, v_2^i)^\top$

$$E(\mathcal{F}) = \sum_{i=1}^{N}\left(\frac{1}{(\mathcal{F}\,\tilde{\mathbf{m}}_1^i)_1^2 + (\mathcal{F}\,\tilde{\mathbf{m}}_1^i)_2^2 + (\mathcal{F}^\top\,\tilde{\mathbf{m}}_2^i)_1^2 + (\mathcal{F}^\top\,\tilde{\mathbf{m}}_2^i)_2^2}\right)\left(\tilde{\mathbf{m}}_2^{i\top} \mathcal{F}\,\tilde{\mathbf{m}}_1^i\right)^2 .$$

Difficult to minimise due to nonlinearity and non-convexity!

RANSAC

- *Question:* How can we make our results even more robust under noise?
- *Idea:* Estimate the fundamental matrix exclusively on correspondence inliers. Such **Random Sampling Consensus (RANSAC)** techniques work as follows:
  1) draw m random samples of with $q = 7/q = 8$ correspondences each (minimal set to compute the fundamental matrix $\mathcal{F}$)
  2) for each sample j compute the corresponding fundamental matrix $\mathcal{F}_j$ (using one of the previously discussed methods)
  3) for each matrix $\mathcal{F}_j$ compute the number of consistent correspondences (the correspondences for which the distance $d^2(\tilde{\mathbf{m}}_2^i, \mathcal{F}_j\, \tilde{\mathbf{m}}_1^i) < \sigma^2$)
  4) keep the matrix $\mathcal{F}_j$ that yields the largest consistent set, i.e. $\mathcal{F} = \mathcal{F}_j$
- *Extension:* Discard outliers and recompute the fundamental matrix
  5) remove inconsistent correspondences with respect to $\mathcal{F}_j$
  6) recompute $\mathcal{F}$ based only on the consistent correspondences

MI A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Important Issues for Using the RANSAC Method

- *Number of Samples:* How much samples have to be drawn to get a useful result?
  - Given a fraction of ϵ outliers, the probability of having at least one good subsample among the m samples is given by

$$p = 1 - [1 - (1-\epsilon)^q]^m \, .$$

  - For a desired probability P of having at least one good subsample one can thus compute the required number of draws given by

$$m = \frac{\log(1-P)}{\log[1-(1-\epsilon)^q]} \, .$$

    For $q = 8$ and $P = 0.99$ and 40% outliers this results in $m = 272$ draws.

- *Scattered Correspondences:* How shall we draw our correspondences?
  - In order to obtain linear independent correspondences they should be picked well-distributed over the whole image domain.

Normalisation and Rank-2 Constraint

- *Problem:* For common image sizes (e.g. 512×512), the first two and the third entry of projective points may have different scales (e.g. $\widetilde{\mathbf{m}} = (512, 512, 1)^\top$). ($\rightarrow$ points with large entries have more influence in the estimation process)
- *Idea:* Consider for all given correspondences the points in the left and the right image separately and **normalise** the corresponding data sets M_1 and M_2 via *(Hartley 1997)*

$$\overline{\mathbf{m}}_j^i = T_j \, \widetilde{\mathbf{m}}_j^i$$

for $i = 1, ..., N$ and $j = 1, 2$. The two transformation matrices T_1 and T_2 read

$$T_j = \begin{pmatrix} s_j & 0 & -s_j \, \mu x_j \\ 0 & s_j & -s_j \, \mu y_j \\ 0 & 0 & 1 \end{pmatrix}$$

where s_j denotes a scaling and μx_j and μy_j denote a shift in x- and y-direction for the two images $j = 1, 2$. How are these parameters determined in practice?

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Choice of Normalisation Parameters

- *Idea:* Chose the parameters for T_1 and T_2 such that the normalised data sets $\overline{M}_1$ and $\overline{M}_2$ have zero mean and the average magnitude of their vectors $\overline{\mathbf{m}}_j^i$ is $\sqrt{3}$.
- *Step 1:* We compute the mean for each data set M_1 and M_2 via the equations

$$\mu x_j = \frac{1}{N} \sum_{i=1}^{N} u_j^i \quad , \quad \mu y_j = \frac{1}{N} \sum_{i=1}^{N} v_j^i$$

- *Step 2:* We compute the scaling as inverse of the average vector magnitude via

$$s_j = \frac{\sqrt{3}}{\frac{1}{N} \sum_{i=1}^{N} \sqrt{(u_j^i - \mu x_j)^2 + (y_j^i - \mu y_j)^2 + 1^2}}$$

How can we integrate this normalisation in the computation of the fundamental matrix? Can we still use the same techniques?

Normalisation and Rank-2 Constraint (3)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Integration of the Normalisation

- *Idea:* The integration of the normalisation is done in three steps
- *Step 1:* We normalise our input data sets M_1 and M_2 using the transformations

$$\overline{\mathbf{m}}_1^{\,i} = T_1 \, \widetilde{\mathbf{m}}_1^i \quad \text{and} \quad \overline{\mathbf{m}}_2^{\,i} = T_2 \, \widetilde{\mathbf{m}}_2^i \, .$$

- *Step 2:* We compute the fundamental matrix $\overline{\mathcal{F}}$ using the normalised input data sets and some **previous technique** based on the epipolar constraint

$$\widetilde{\mathbf{m}}_2^{i\top} \, \mathcal{F} \, \widetilde{\mathbf{m}}_1^i = \overline{\mathbf{m}}_2^{\,i\top} \, T_2^{-\top} \, \mathcal{F} \, T_1^{-1} \, \overline{\mathbf{m}}_1^{\,i} = \overline{\mathbf{m}}_2^{\,i\top} \, \overline{\mathcal{F}} \, \overline{\mathbf{m}}_1^{\,i} \, .$$

- *Step 3:* We recover the correct fundamental matrix $\mathcal{F}$ from the result $\overline{\mathcal{F}}$ for the normalised input data sets via the final transformation

$$\mathcal{F} = T_2^\top \, \overline{\mathcal{F}} \, T_1 \, .$$

Normalisation and Rank-2 Constraint (4)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

How Can We Enforce the Rank-2 Constraint after the Computation?

- *Idea:* Enforce rank-2 by setting the smallest singular value of $\mathcal{F}$ zero
- *Example:* Let us consider the singular value decomposition of $\mathcal{F}$, where the singular values are assumed to be ordered, i.e. $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$

$$\mathcal{F} = UDV^T = U \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} V^T \, .$$

  Here, U and V are orthogonal matrices containing left and right eigenvectors.

- *Projection:* Setting the smallest singular value of $\mathcal{F}$ zero, i.e. $\sigma_3 := 0$, we obtain

$$\mathcal{F}' = U \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T$$

  which is the closest rank-2 matrix to original matrix $\mathcal{F}$ in the Frobenius norm.

Error Measures for the Fundamental Matrix

- *Question:* Given an estimated fundamental matrix $\mathcal{F}_\mathrm{e}$ and the corresponding ground truth $\mathcal{F}_\mathrm{e}$, how can we judge the quality of the estimated matrix?
- *Error Measure 1:* As simple measure is the **Frobenius norm** of the difference of both normalised matrices given by

$$\Delta_\mathrm{F}(\mathcal{F}_\mathrm{e}, \mathcal{F}_\mathrm{t}) = \min\left(\left| \frac{\mathcal{F}_\mathrm{t}}{|\mathcal{F}_\mathrm{t}|_F} + \frac{\mathcal{F}_\mathrm{e}}{|\mathcal{F}_\mathrm{e}|_F} \right|_F , \left| \frac{\mathcal{F}_\mathrm{t}}{|\mathcal{F}_\mathrm{t}|_F} - \frac{\mathcal{F}_\mathrm{e}}{|\mathcal{F}_\mathrm{e}|_F} \right|_F \right) .$$

  Since $\mathcal{F}_\mathrm{t}$ is only defined up to a (possibly negative) scale, one must consider both the error for the positive and negative version of the normalised variant of $\mathcal{F}_\mathrm{e}$.
- *Drawbacks*: This measure is not frequently used in practice.
  - no geometrical interpretation of the obtained error
  - hardly used in practice, since a small value for the error may still result in completely different epipolar geometry of the camera system

MIA 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Can We Find Some Error Measure that Allows for a Geometric Interpretation?

- *Error Measure 2:* A much more complicated error measure is the **symmetric distance error measure** that consists of five different steps:
  1) Select random point $\widetilde{\mathbf{m}}_1$ in the left image and determine its corresponding epipolar lines $\mathbf{l}_{2,\mathrm{e}} = \mathcal{F}_\mathrm{e}\,\widetilde{\mathbf{m}}_1$ and $\mathbf{l}_{2,\mathrm{t}} = \mathcal{F}_\mathrm{t}\,\widetilde{\mathbf{m}}_1$ in the second image
  2) Select random point $\widetilde{\mathbf{m}}_{2,\mathrm{e}}$ on the epipolar line $\mathbf{l}_{2,\mathrm{e}}$ and random point $\widetilde{\mathbf{m}}_{2,\mathrm{t}}$ on the epipolar line $\mathbf{l}_{2,\mathrm{t}}$
  3) Determine the corresponding epipolar lines $\mathbf{l}_{1,\mathrm{t}} = \mathcal{F}_\mathrm{t}^\top \widetilde{\mathbf{m}}_{2,\mathrm{e}}$ and $\mathbf{l}_{1,\mathrm{e}} = \mathcal{F}_\mathrm{e}^\top \widetilde{\mathbf{m}}_{2,\mathrm{t}}$ in the first image (note that here the matrices $\mathcal{F}_\mathrm{e}$ and $\mathcal{F}_\mathrm{r}$ are exchanged)
  4) Compute the following four symmetrical distances and average them

$$d_i = \frac{1}{4}\Big(\underbrace{d_\mathrm{NL}(\widetilde{\mathbf{m}}_1, \mathbf{l}_{1,\mathrm{t}})}_{d_{1,\mathrm{t}}} + \underbrace{d_\mathrm{NL}(\widetilde{\mathbf{m}}_1, \mathbf{l}_{1,\mathrm{e}})}_{d_{1,\mathrm{e}}} + \underbrace{d_\mathrm{NL}(\widetilde{\mathbf{m}}_{2,\mathrm{e}}, \mathbf{l}_{2,\mathrm{t}})}_{d_{2,\mathrm{t}}} + \underbrace{d_\mathrm{NL}(\widetilde{\mathbf{m}}_{2,\mathrm{t}}, \mathbf{l}_{2,\mathrm{e}})}_{d_{2,\mathrm{e}}} \Big) .$$

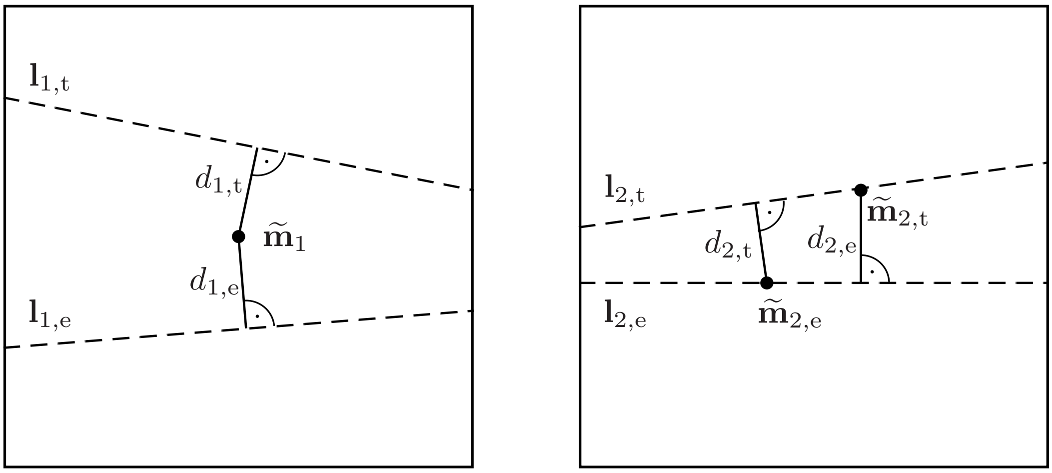
  5) Repeat steps 1-4 N times and compute the final error measure **in pixels** as

$$\Delta_\mathrm{d}(\mathcal{F}_\mathrm{e}, \mathcal{F}_\mathrm{t}) = \frac{1}{N}\sum_{i=1}^{N} d_i .$$

Error Measures for the Fundamental Matrix (3)

Illustration of the Symmetrical Distance Error Measure

- *Illustration:* Computation of the four distances $d_{1,\mathrm{t}}$, $d_{1,\mathrm{e}}$, $d_{2,\mathrm{t}}$ and $d_{2,\mathrm{e}}$



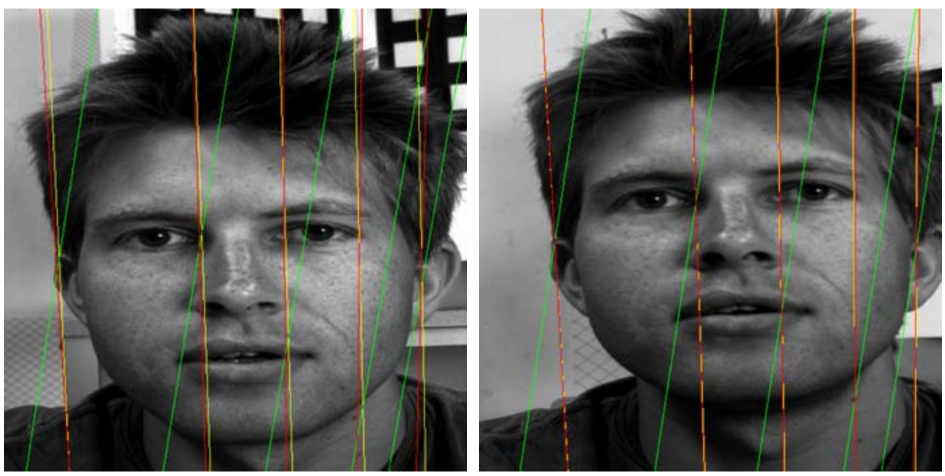
The symmetrical distance error measure. *Author:* M. Mainberger.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Error Measures for the Fundamental Matrix (4)

Symmetric Distance Measure vs. Normalised Frobenius Error

- In most cases the normalised Frobenius error is not a very good indicator



Left: Left image of the Herve stereo pair. **Right:** Right image. For some points epipolar lines are shown. Ground truth (yellow), symmetric distance measure Δ_{d} (red) and normalised Frobenius Error Δ_{F} (green). *Author:* M. Mainberger.

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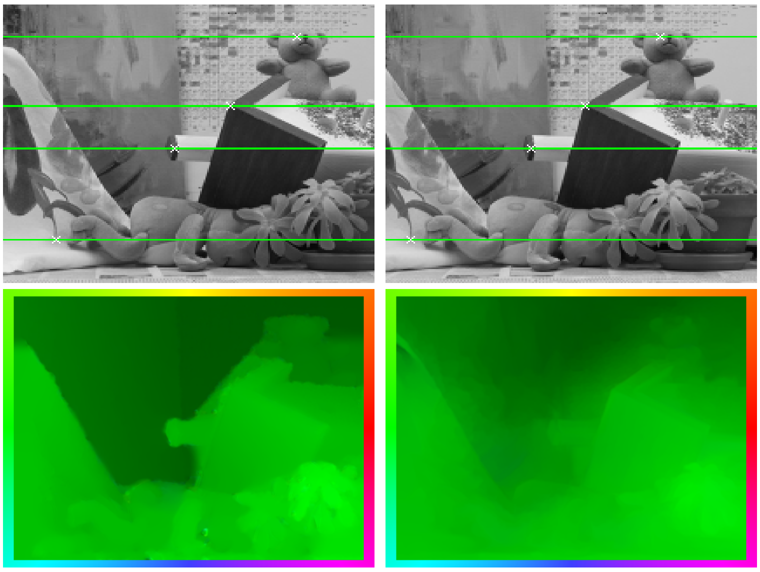
Normalised Least Squares Estimation from Variational Optic Flow Matches



Ortho-parallel camera setup. **Top Row:** Left and right frame of the Teddy pair (Middlebury benchmark) with epipolar lines shown for the ground truth (black), Brox *et al.* (green) and Nagel/Enkelmann (red). **Bottom Row:** Flow field of Brox *et al.* (left) and Nagel/Enkelmann (right). *Author:* M. Mainberger.

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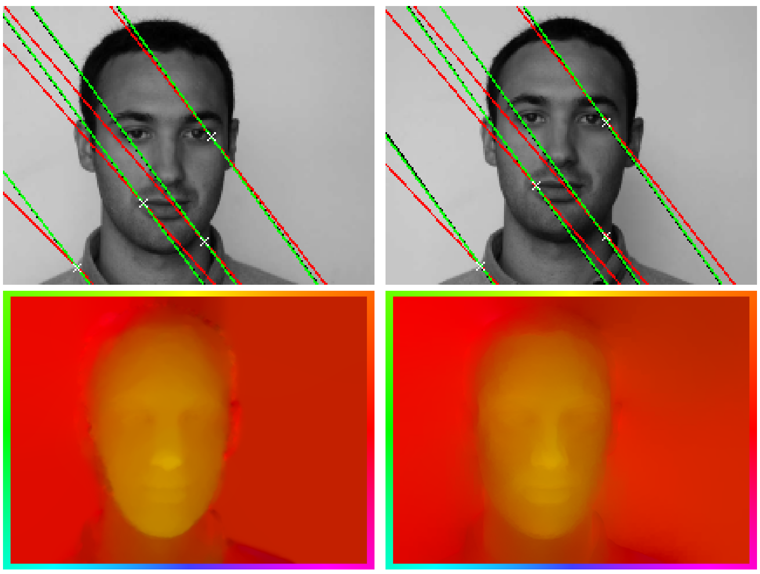
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Normalised Least Squares Estimation from Variational Optic Flow Matches



Converging camera setup. **Top Row:** Left and right frame of the Javier pair with epipolar lines shown for the ground truth (black), Brox *et al.* (green) and Nagel/Enkelmann (red). **Bottom Row:** Flow field of Brox *et al.* (left) and Nagel/Enkelmann (right). *Author:* M. Mainberger.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Normalised Least Squares Estimation from Variational Optic Flow Matches

- *Ortho-Parallel Camera Setup:* Comparison to the ground truth for Teddy

	OLS-SIFT	RANSAC-OLS-SIFT	OLS-Nagel/Enkelmann	OLS-Brox et al.
Δ_d	95.014	0.804	**0.302**	**0.109**

- *Converging Camera Setup:* Comparison to the ground truth for Javier

	OLS-SIFT	RANSAC-OLS-SIFT	OLS-Nagel/Enkelmann	OLS-Brox et al.
Δ_d	151.915	10.202	9.736	**1.441**

- *Observations:* One can observe the following tendencies
  - ortho-parallel results are more accurate than converging results
  - variational techniques allow robust results even with non-robust estimators
  - features based techniques require RANSAC due to outliers

Summary

- The estimation of the fundamental matrix requires
  - at least 7 correspondences (respecting the cubic rank-2 constraint)
  - at least 8 correspondences (enforcing the rank-2 constraint a posteriori)
- Least squares fits and total least squares fits
  - are easy to implement and offer a fast performance (simple minimisation)
  - are highly sensitive w.r.t. outliers (require accurate correspondences)
- Geometric methods
  - can be interpreted geometrically (e.g. as distances to epipolar lines)
  - are difficult to optimise (nonlinear, nonconvex)
- M-estimators can improve the performance by downweighting outliers
- RANSAC methods use small subsets and thus discard outliers completely
- Good error measures for the fundamental matrix are difficult to find

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  (*more theoretical book on projective geometry and stereo reconstruction)*
- E. Trucco, A. Verri:
  Introductory Techniques for 3-D Computer Vision.
  Prentice Hill, 1998.
  (*introduction to computer vision with chapters on stereo)*

Assignment 8

Theoretical Exercise 1 (Fundamental Matrix)

Let two input images f_1 and f_2 be given together with the fundamental matrix F. If you downsample both images to half of their original resolution, how does the fundamental matrix change?

Theoretical Exercise 2 (Stereo Reconstruction)

Let us assume that we have to cameras C_1 and C_2. The extrinsic and intrinsic parameters of both cameras are given in form of the matrices

$$A_1^{\text{int}} = \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_2^{\text{int}} = \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_1^{\text{ext}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_2^{\text{ext}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -4 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Moreover, the focal length of both cameras is given by $f_1 = f_2 = 1$.

MIA 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Assignment 8

Theoretical Exercise 2 (Stereo Reconstruction - continued)

(a) Compute the inverse of A_2^{int} and A_2^{ext}. They will be required for the tasks (b) and (c).

(b) Let us assume that we have found the following correspondence

$$\begin{aligned} \mathbf{x_1} &= (10.5, 0.5)^\top \\ \mathbf{x_2} &= (9.5, \frac{1}{\sqrt{2}})^\top \end{aligned}$$

Here, $\mathbf{x_1}$ and $\mathbf{x_2}$ denote a point in pixel locations in the first and in the second frame, respectively. Compute the corresponding optical rays in the 3-D coordinate systems of the cameras.
(*Hint: Formulate the points in 2-D homogeneous coordinates, compensate for the intrinsic parameters and use your knowledge about the relation between homogeneous 2-D and Euclidean 3-D coordinates to obtain the corresponding rays.)*

(c) Compute the corresponding optical rays in the 3-D Euclidean world coordinate system.
(*Hint: Formulate the rays in 3-D homogeneous coordinates, compensate for the extrinsic parameters and use your knowledge on the conversion from 3-D homogeneous coordinates to 3-D Euclidean coordinates.)*

(d) Intersect these rays to recover the 3-D world coordinates of the 3-D point that was projected on both image planes. What is the depth of this point (z-coordinate)?