8.1 Fundamental Matrix

We have the epipolar constraint:

$$\tilde{m}_2 F \tilde{m}_1 = 0 = \begin{pmatrix} u_2 \\ v_2 \\ 1 \end{pmatrix}^\top F \begin{pmatrix} u_1 \\ v_1 \\ 1 \end{pmatrix}.$$

Reformulating this, we get

$$\begin{aligned}
0 &= \begin{pmatrix} 2\hat{u}_2 \\ 2\hat{v}_2 \\ 1 \end{pmatrix}^\top F \begin{pmatrix} 2\hat{u}_1 \\ 2\hat{v}_1 \\ 1 \end{pmatrix} \\
&= \left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{u}_2 \\ \hat{v}_2 \\ 1 \end{pmatrix} \right)^\top F \left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{v}_1 \\ 1 \end{pmatrix} \right) \\
&= \begin{pmatrix} \hat{u}_2 \\ \hat{v}_2 \\ 1 \end{pmatrix}^\top \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^\top F \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{v}_1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \hat{u}_2 \\ \hat{v}_2 \\ 1 \end{pmatrix}^\top \underbrace{\begin{pmatrix} 4f_{11} & 4f_{12} & 2f_{13} \\ 4f_{21} & 4f_{22} & 2f_{23} \\ 2f_{31} & 2f_{32} & f_{33} \end{pmatrix}}_{\hat{F}} \begin{pmatrix} \hat{u}_1 \\ \hat{v}_1 \\ 1 \end{pmatrix}
\end{aligned}$$

We also know

$$F = A_{int,2}^{-\top} [t]_\times R A_{int,1}^{-1}$$

with

$$A_{int} = \begin{pmatrix} k_u & -k_u \cot\varphi & u_0 \\ 0 & k_v / \sin\varphi & v_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark: k_u and k_v are measured in $\frac{pix}{m}$.
With $\hat{k_u} = \frac{1}{2}k_u$ and $\hat{k_v} = \frac{1}{2}k_v$, we get

$$\begin{aligned}
\hat{A_{int}} &= \begin{pmatrix} \hat{k_u} & -\hat{k_u}\cot\varphi & \hat{u_0} \\ 0 & \hat{k_v}/\sin\varphi & \hat{v_0} \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}k_u & -\frac{1}{2}k_u\cot\varphi & \frac{1}{2}u_0 \\ 0 & \frac{1}{2}k_v/\sin\varphi & \frac{1}{2}v_0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} A_{int} \\
\hat{A_{int}}^{-1} &= \left(\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} A_{int} \right)^{-1} \\
&= A_{int}^{-1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

8.2 Stereo Reconstruction

a. We have

$$\begin{aligned}
A_1^{int} = A_2^{int} &= \left(\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \\
\left(A_1^{int}\right)^{-1} = \left(A_2^{int}\right)^{-1} &= \left(\begin{array}{cc|c} 1 & 0 & -10 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \\
A_1^{ext} &= \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \\
A_2^{ext} &= \left(\begin{array}{ccc|c} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -4 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)
\end{aligned}$$

In general, it holds:

$$
\begin{aligned}
A &= \left(\begin{array}{c|c} R^{3\times 3} & t^3 \\ \hline {0^3}^\top & 1 \end{array}\right) \\
A^{-1} &= \left(\begin{array}{c|c} R^{-1} & R^{-1}t \\ \hline {0^3}^\top & 1 \end{array}\right)
\end{aligned}
$$

It follows from this fact

$$
\left(A_2^{ext}\right)^{-1} = \left(\begin{array}{ccc|c} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{4}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{4}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)
$$

b. The "roadmap" looks as follows:

3-D world coordinates	
↓	extrinsic parameters
3-D camera coordinates	
↓	projection step
2-D image coordinates	
↓	intrinsic parameters
2-D pixel coordinates	

Given 2-D Euclidean coordinates

$$
\begin{aligned}
x_1 &= (10.5, 0.5)^\top \\
x_2 &= \left(9.5, \frac{1}{\sqrt{2}}\right)^\top
\end{aligned}
$$

we have to go to 2-D homogeneous coordinates

$$
\begin{aligned}
\hat{x_1} &= (10.5, 0.5, 1)^\top \\
\hat{x_2} &= \left(9.5, \frac{1}{\sqrt{2}}, 1\right)^\top
\end{aligned}
$$

$$
\begin{aligned}
\left(A_1^{int}\right)^{-1}\hat{x}_1 &= \begin{pmatrix} 1 & 0 & -10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 10.5 \\ 0.5 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \end{pmatrix} \\
\left(A_2^{int}\right)^{-1}\hat{x}_2 &= \begin{pmatrix} 1 & 0 & -10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 9.5 \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -0.5 \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}
\end{aligned}
$$

Let's go to the 3-D camera coordinates ($\rightarrow$ 3-D Euclidean coordinates):

$$
\begin{aligned}
X_1 &= \lambda_1 \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \end{pmatrix} \quad , \lambda_1 \in \mathbb{R} \qquad \leftarrow \text{seen from first camera} \\
X_2 &= \lambda_2 \begin{pmatrix} -0.5 \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} \quad , \lambda_2 \in \mathbb{R} \qquad \leftarrow \text{seen from the second camera}
\end{aligned}
$$

Remark: Do not intersect these two points! They live in different coordinate systems!

c. Let's go to 3-D homogeneous coordinates:

$$
\begin{aligned}
\hat{X}_1 &= (0.5\lambda_1, 0.5\lambda_1, \lambda_1, 1)^\top \\
\hat{X}_2 &= \left(-0.5\lambda_2, \frac{1}{\sqrt{2}}\lambda_2, \lambda_2, 1\right)^\top
\end{aligned}
$$

Now, we can go to 3-D world coordinates:

$$\left(A_1^{ext}\right)^{-1}\hat{X}_1 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0.5\lambda_1\\0.5\lambda_1\\1\end{pmatrix} = \begin{pmatrix}0.5\lambda_1\\0.5\lambda_1\\\lambda_1\\1\end{pmatrix}$$

$$\left(A_2^{ext}\right)^{-1}\hat{X}_2 = \begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&\frac{4}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0&-\frac{4}{\sqrt{2}}\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}-0.5\lambda_2\\\frac{1}{\sqrt{2}}\lambda_2\\\lambda_2\\1\end{pmatrix} = \begin{pmatrix}\frac{2-\sqrt{2}}{4}\lambda_2+2\sqrt{2}\\\frac{2+\sqrt{2}}{4}\lambda_2-2\sqrt{2}\\\lambda_2\\1\end{pmatrix}$$

d. 3-D Euclidean is now

$$L_1 = \lambda_1\begin{pmatrix}0.5\\0.5\\1\end{pmatrix} \quad ,\lambda_1\in\mathbb{R}$$

$$L_2 = \lambda_2\begin{pmatrix}\frac{2-\sqrt{2}}{4}\\\frac{2+\sqrt{2}}{4}\\1\end{pmatrix}+\begin{pmatrix}2\sqrt{2}\\-2\sqrt{2}\\0\end{pmatrix} \quad ,\lambda_2\in\mathbb{R}$$

These two lines can now be intersected since they live in the same coordinate system:

$$\lambda_1\begin{pmatrix}0.5\\0.5\\1\end{pmatrix} = \lambda_2\begin{pmatrix}\frac{2-\sqrt{2}}{4}\\\frac{2+\sqrt{2}}{4}\\1\end{pmatrix}+\begin{pmatrix}2\sqrt{2}\\-2\sqrt{2}\\0\end{pmatrix} \quad \leftarrow \text{ tells } \lambda_1=\lambda_2$$

$$\lambda_1\begin{pmatrix}\frac{\sqrt{2}}{4}\\-\frac{\sqrt{2}}{4}\\0\end{pmatrix} = \begin{pmatrix}2\sqrt{2}\\-2\sqrt{2}\\0\end{pmatrix}$$

$$\Rightarrow \lambda_1=\lambda_2 = 8$$

$M_w = (4,4,8)^\top$: The depth of this point is 8.