6.1 Warping and SOR

a. If we rearrange the two equations, we get

$$\begin{aligned}\alpha\Delta u^k - J_{13}^k &= J_{11}^k du^k + J_{12}^k dv^k - \alpha\Delta du^k \\ \alpha\Delta v^k - J_{23}^k &= J_{12}^k du^k + J_{22}^k dv^k - \alpha\Delta dv^k\end{aligned}$$

$$\begin{aligned}0 &= [J_{11}^k]_{i,j} du_{i,j}^k + [J_{12}^k]_{i,j} dv_{i,j}^k + [J_{13}^k]_{i,j} - \alpha \sum\sum \frac{du_{\tilde{i},\tilde{j}}^k - du_{i,j}^k}{h_l^2} \\ &\quad -\alpha \sum\sum \frac{u_{\tilde{i},\tilde{j}}^k - u_{i,j}^k}{h_l^2} \\ 0 &= [J_{12}^k]_{i,j} du_{i,j}^k + [J_{22}^k]_{i,j} dv_{i,j}^k + [J_{23}^k]_{i,j} - \alpha \sum\sum \frac{dv_{\tilde{i},\tilde{j}}^k - dv_{i,j}^k}{h_l^2} \\ &\quad -\alpha \sum\sum \frac{v_{\tilde{i},\tilde{j}}^k - v_{i,j}^k}{h_l^2}\end{aligned}$$

where the double-summation has the following arguments (they are omitted due to limited space):

$$\sum_{l \in x,y} \sum_{(\tilde{i},\tilde{j}) \in \mathcal{N}_l(i,j)} .$$

From this we have to go to an equation system of the form $A\mathbf{x} = \mathbf{b}$. Let's define the following:

$$\begin{aligned}\mathbf{x} &:= \begin{pmatrix} du^k \\ du^k \\ du^k \\ dv^k \\ dv^k \\ dv^k \end{pmatrix} \\ \mathbf{b} &:= \begin{pmatrix} J_{13}^k \\ J_{13}^k \\ J_{13}^k \\ J_{23}^k \\ J_{23}^k \\ J_{23}^k \end{pmatrix} - \alpha D \begin{pmatrix} u^k \\ u^k \\ u^k \\ v^k \\ v^k \\ v^k \end{pmatrix}\end{aligned}$$

$$
A := \begin{pmatrix}
J_{11}^k & & & J_{12}^k & & \\
& J_{11}^k & & & J_{12}^k & \\
& & J_{11}^k & & & J_{12}^k \\
J_{12}^k & & & J_{22}^k & & \\
& J_{12}^k & & & J_{22}^k & \\
& & J_{12}^k & & & J_{22}^k
\end{pmatrix}
+\alpha \underbrace{\begin{pmatrix}
1 & -1 & & & & \\
-1 & 2 & -1 & & & \\
& -1 & 1 & & & \\
& & & 1 & -1 & \\
& & & -1 & 2 & -1 \\
& & & & -1 & 1
\end{pmatrix}}_{=:D}
$$

b. The structure of the double-summation is slightly changed here, since one sometimes needs $\mathcal{N}_l^-$ (newly computed) and $\mathcal{N}_l^+$ (old ones). First, we compute the Gauß-Seidel method. After that, the step to SOR is very easy.

$$
\begin{aligned}
\tilde{du}_{i,j}^{m+1,k} &= \frac{\alpha \sum\sum \frac{u_{\tilde{i},\tilde{j}}^k - u_{i,j}^k}{h_l^2} - [J_{13}^k]_{i,j} - \left([J_{12}^k]_{i,j} dv_{i,j}^{m,k} - \alpha \sum\sum_{\mathcal{N}_l^-} \frac{du_{\tilde{i},\tilde{j}}^{m+1,k}}{h_l^2} - \alpha \sum\sum_{\mathcal{N}_l^+} \frac{du_{\tilde{i},\tilde{j}}^{m,k}}{h_l^2} \right)}{[J_{11}]_{i,j} + \alpha \sum\sum \frac{1}{h_l^2}} \\
\tilde{dv}_{i,j}^{m+1,k} &= \frac{\alpha \sum\sum \frac{v_{\tilde{i},\tilde{j}}^k - v_{i,j}^k}{h_l^2} - [J_{23}^k]_{i,j} - \left([J_{12}^k]_{i,j} du_{i,j}^{m+1,k} - \alpha \sum\sum_{\mathcal{N}_l^-} \frac{dv_{\tilde{i},\tilde{j}}^{m+1,k}}{h_l^2} - \alpha \sum\sum_{\mathcal{N}_l^+} \frac{dv_{\tilde{i},\tilde{j}}^{m,k}}{h_l^2} \right)}{[J_{22}]_{i,j} + \alpha \sum\sum \frac{1}{h_l^2}}
\end{aligned}
$$

where m is the iteration step and k the warping level.
Remark: In the formula for $\tilde{dv}_{i,j}^{m+1,k}$ appears $du_{i,j}$ with $m+1, k$. This is due to the fact that du is already computed, assuming the ordering $du \rightarrow dv$ for the computation.
Let's state the SOR step:

$$
\begin{aligned}
du_{i,j}^{m+1,k} &= (1-\omega) du_{i,j}^{m,k} + \omega \tilde{du}_{i,j}^{m+1,k} \\
dv_{i,j}^{m+1,k} &= (1-\omega) dv_{i,j}^{m,k} + \omega \tilde{dv}_{i,j}^{m+1,k}
\end{aligned}
$$

6.2 Photometric Invariants and Robustification

a. Let's first give an overview over the different properties where the following assumptions are considered:

① global multiplicative illumination changes

② shadow and shading (local multiplicative changes)

③ highlights and specular reflections (local additive / local multiplicative changes)

	①	②	③
p_1	✘	✘	✘
p_2	✘	✘	✘
p_3	✔	✔	✘
p_4	✔	✔	✔
p_5	✔	✘	✘
p_6	✔	✘	✘

For the decision, what classes do apply, we consider Lecture 5, slide 21.

1.

$$\begin{aligned} p_1 &= R - 3B + G \\ &= e\left(m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x})\right) - 3e\left(m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{B}_b(\mathbf{x})\right) \\ &\quad + e\left(m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{G}_b(\mathbf{x}\right) \\ &= e\left(-m_i(\mathbf{x})w_i + m_b(\mathbf{x})(\hat{R}_b(\mathbf{x}) - 3\hat{B}_b(\mathbf{x}) + \hat{G}_b(\mathbf{x}))\right) \end{aligned}$$

2. Here we have an invariance under local additive changes only.

$$\begin{aligned} p_2 &= R^2 + B^2 - 2BR = (R - B)^2 \\ &= \left[e\left(m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x}) - m_i(\mathbf{x})w_i - m_b(\mathbf{x})\hat{B}_b(\mathbf{x})\right)\right]^2 \\ &= \left[em_b(\mathbf{x})(\hat{R}_b(\mathbf{x}) - \hat{B}_b(\mathbf{x}))\right]^2 \\ &= e^2 m_b(\mathbf{x})^2(\hat{R}_b(\mathbf{x}) - \hat{B}_b(\mathbf{x}))^2 \end{aligned}$$

3.

$$
\begin{aligned}
p_3 &= \frac{R-B}{R+G} \\
&= \frac{e\Big(m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x}) - m_i(\mathbf{x})w_i - m_b(\mathbf{x})\hat{B}_b(\mathbf{x})\Big)}{e\Big(m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x}) + m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{G}_b(\mathbf{x})\Big)} \\
&= \frac{m_b(\mathbf{x})\left(\hat{R}_b(\mathbf{x}) - \hat{B}_b(\mathbf{x})\right)}{2m_i(\mathbf{x})w_i + m_b(\mathbf{x})\left(\hat{R}_b(\mathbf{x}) + \hat{B}_b(\mathbf{x})\right)} \\
&= \frac{\hat{R}_b(\mathbf{x}) - \hat{B}_b(\mathbf{x})}{\hat{R}_b(\mathbf{x}) + \hat{B}_b(\mathbf{x})} \quad \text{if } m_i(\mathbf{x}) = 0.
\end{aligned}
$$

4.

$$
\begin{aligned}
p_4 &= 2\frac{BG+RG}{B^2-R^2} - \frac{B+R}{B-R} \\
&= 2\frac{G(B+R)}{(B-R)(B+R)} - \frac{B+R}{B-R} = \frac{2G-B-R}{B-R} \\
&= \frac{m_b(\mathbf{x})\left(2\hat{G}_b(\mathbf{x}) - \hat{B}_b(\mathbf{x}) - \hat{R}_b(\mathbf{x})\right)}{m_b(\mathbf{x})\left(\hat{B}_b(\mathbf{x}) - \hat{R}_b(\mathbf{x})\right)} \\
&= \frac{2\hat{G}_b(\mathbf{x}) - \hat{B}_b(\mathbf{x}) - \hat{R}_b(\mathbf{x})}{\hat{B}_b(\mathbf{x}) - \hat{R}_b(\mathbf{x})}
\end{aligned}
$$

5. Let's consider the following fact

$$
\ln(\underbrace{c}_{\text{factor}} B)_x = (\underbrace{\ln c}_{\text{constant}} + \ln B)_x = (\ln B)_x
$$

$$
\begin{aligned}
p_5 &= (\ln B)_x + (\ln R)_x \\
&= \frac{1}{\left[e(m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{B}_b(\mathbf{x}))\right]_x} + \frac{1}{\left[e(m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x}))\right]_x}
\end{aligned}
$$

6. Invariant under global additive only.

$$
\begin{aligned}
p_6 &= \ln B_x + \ln R_x = \ln(B_x \cdot R_x) \\
&= \ln\left[\Big(e(m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{B}_b(\mathbf{x}))\Big)_x \cdot \Big(e(m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x}))\Big)_x\right]
\end{aligned}
$$

b. We consider here

$$\Psi(s^2) = \sqrt{s^2 + \varepsilon^2} \quad \text{with } \varepsilon = 10^{-3}.$$

Let's give the joint robustification

$$\begin{aligned} E_{joint}(u,v) &= \int_\Omega \Psi\left(\sum_{i=1}^{6} \left(p_i(x+u,y+v,t+1) - p_i(x,y,t)\right)^2\right) \\ &\quad +\alpha\left(|\nabla u|^2 + |\nabla v|^2\right)\, \mathrm{d}x\mathrm{d}y \end{aligned}$$

c. Joint robustification was given in part **b.**. Separate robustification is given by

$$\begin{aligned} E_{sep}(u,v) &= \int_\Omega \sum_{i=1}^{6} \Psi\left(\left(p_i(x+u,y+v,t+1) - p_i(x,y,t)\right)^2\right) \\ &\quad +\alpha\left(|\nabla u|^2 + |\nabla v|^2\right)\, \mathrm{d}x\mathrm{d}y \end{aligned}$$

The Euler-Lagrange equations that have to be satisfied are

$$\begin{aligned} 0 &= F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} \\ 0 &= F_v - \frac{\partial}{\partial x} F_{v_x} - \frac{\partial}{\partial y} F_{v_y} \end{aligned}$$

- joint robustification:

$$\begin{aligned} F &= \Psi\left(\sum_{i=1}^{6} \left(p_i(x+u,y+v,t+1) - p_i(x,y,t)\right)^2\right) \\ &\quad +\alpha\left(|\nabla u|^2 + |\nabla v|^2\right) \\ F_u &= 2 \cdot \Psi'(...) \\ &\quad \cdot \sum_{i=1}^{6} (p_i(x+u,y+v,t+1) - p_i(x,y,t)) \\ &\qquad \cdot p_{i_x}(x+u,y+v,t+1) \\ F_v &= 2 \cdot \Psi'(...) \\ &\quad \cdot \sum_{i=1}^{6} (p_i(x+u,y+v,t+1) - p_i(x,y,t)) \\ &\qquad \cdot p_{i_y}(x+u,y+v,t+1) \\ F_{u_x} &= 2\alpha u_x \\ F_{u_y} &= 2\alpha u_y \\ F_{v_x} &= 2\alpha v_x \\ F_{v_y} &= 2\alpha v_y \end{aligned}$$

The Euler-Lagrange equations now read as (divided by 2):

$$
\begin{aligned}
0 &= \Psi'(...)\Big(\sum_{i=1}^{6}(p_i(x+u,y+v,t+1)-p_i(x,y,t))p_{i_x}(x+u,y+v,t+1)\Big)-\alpha\Delta u\\
0 &= \Psi'(...)\Big(\sum_{i=1}^{6}(p_i(x+u,y+v,t+1)-p_i(x,y,t))p_{i_y}(x+u,y+v,t+1)\Big)-\alpha\Delta v
\end{aligned}
$$

- separate robustification:

$$
\begin{aligned}
F &= \sum_{i=1}^{6}\Psi\Big(\big(p_i(x+u,y+v,t+1)-p_i(x,y,t)\big)^2\Big)\\
&\quad +\alpha\left(|\nabla u|^2+|\nabla v|^2\right)\\
F_u &= 2\cdot\sum_{i=1}^{6}\Psi'(...)\cdot(p_i(x+u,y+v,t+1)-p_i(x,y,t))\\
&\quad \cdot p_{i_x}(x+u,y+v,t+1)\\
F_v &= 2\cdot\sum_{i=1}^{6}\Psi'(...)\cdot(p_i(x+u,y+v,t+1)-p_i(x,y,t))\\
&\quad \cdot p_{i_y}(x+u,y+v,t+1)\\
F_{u_x} &= 2\alpha u_x\\
F_{u_y} &= 2\alpha u_y\\
F_{v_x} &= 2\alpha v_x\\
F_{v_y} &= 2\alpha v_y
\end{aligned}
$$

The Euler-Lagrange equations read as (again divided by 2):

$$
\begin{aligned}
0 &= \sum_{i=1}^{6}\Big(\Psi'(...)(p_i(x+u,y+v,t+1)-p_i(x,y,t))\cdot p_{i_x}(x+u,y+v,t+1)\Big)-\alpha\Delta u\\
0 &= \sum_{i=1}^{6}\Big(\Psi'(...)(p_i(x+u,y+v,t+1)-p_i(x,y,t))\cdot p_{i_y}(x+u,y+v,t+1)\Big)-\alpha\Delta v
\end{aligned}
$$