8.1 Fundamental Matrix

We know from the lecture that the epipolar constraint is defined as follows:

$$\tilde{m}_2 F \tilde{m}_1 = 0 = \begin{pmatrix} u_2 \\ v_2 \\ 1 \end{pmatrix}^\top F \begin{pmatrix} u_1 \\ v_1 \\ 1 \end{pmatrix}.$$

Let us assume we know the Fundamental matrix for the original (fine) resolution. Now we change the resolution that the number of pixels is halved in each direction. Consequently, the pixel sizes are twice as large in each direction. If we now have two pixels $(\hat{u}_1, \hat{v}_1)$ and $(\hat{u}_2, \hat{v}_2)$ given in the images with the coarser resolution, their original coordinates in the fine resolution images can be calculated by multiplying the coarse grid coordinates with a factor two. Thus we can use the original Fundamental matrix and obtain:

$$\begin{aligned}
0 &= \begin{pmatrix} 2\hat{u}_2 \\ 2\hat{v}_2 \\ 1 \end{pmatrix}^\top F \begin{pmatrix} 2\hat{u}_1 \\ 2\hat{v}_1 \\ 1 \end{pmatrix} \\
&= \left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{u}_2 \\ \hat{v}_2 \\ 1 \end{pmatrix} \right)^\top F \left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{v}_1 \\ 1 \end{pmatrix} \right) \\
&= \begin{pmatrix} \hat{u}_2 \\ \hat{v}_2 \\ 1 \end{pmatrix}^\top \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^\top F \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{v}_1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \hat{u}_2 \\ \hat{v}_2 \\ 1 \end{pmatrix}^\top \underbrace{\begin{pmatrix} 4f_{11} & 4f_{12} & 2f_{13} \\ 4f_{21} & 4f_{22} & 2f_{23} \\ 2f_{31} & 2f_{32} & f_{33} \end{pmatrix}}_{\hat{F}} \begin{pmatrix} \hat{u}_1 \\ \hat{v}_1 \\ 1 \end{pmatrix}
\end{aligned}$$

We can see that some of the entries of the original Fundamental matrix are scaled by the inverse or the squared of the inverse of the downsampling factor which was 0.5. A second possibility to derive the Fundamental matrix for the coarse resolution is to modify the intrisic matrices that are contained in the original Fundamental matrix:

$$F = A_{int,2}^{-\top} [t]_\times R A_{int,1}^{-1}$$

where each of the two intrinsic matrices reads:

$$A_{int} = \begin{pmatrix} k_u & -k_u \cot\varphi & u_0 \\ 0 & k_v / \sin\varphi & v_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark: k_u and k_v are measured in $\frac{pix}{distance\ unit}$ with the distance unit being for instance 1m or 1cm.

Since on the coarse grid a pixel has twice the size than on the coarse grid, one needs less pixel to cover the same area (or distance in each of the two directions). Knowing this, we can set $\hat{k_u} = \frac{1}{2}k_u$ and $\hat{k_v} = \frac{1}{2}k_v$. Consequently, we obtain our new intrinsic matrices

$$\begin{aligned}
\hat{A_{int}} &= \begin{pmatrix} \hat{k_u} & -\hat{k_u}\cot\varphi & \hat{u_0} \\ 0 & \hat{k_v}/\sin\varphi & \hat{v_0} \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}k_u & -\frac{1}{2}k_u\cot\varphi & \frac{1}{2}u_0 \\ 0 & \frac{1}{2}k_v/\sin\varphi & \frac{1}{2}v_0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} A_{int}
\end{aligned}$$

which results in the inverse intrinsic matrices.

$$\begin{aligned}
\hat{A_{int}}^{-1} &= \left(\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} A_{int} \right)^{-1} \\
&= A_{int}^{-1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} .
\end{aligned}$$

Replacing the intrinsic matrices for the fine resolutions by those for the coarse resolution one obtains

$$\begin{aligned}
0 &= \begin{pmatrix} \hat{u_2} \\ \hat{v_2} \\ 1 \end{pmatrix}^\top \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^\top A_{int,2}^{-\top}[t]_\times R A_{int,1}^{-1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{u_1} \\ \hat{v_1} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \hat{u_2} \\ \hat{v_2} \\ 1 \end{pmatrix}^\top \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^\top F \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{u_1} \\ \hat{v_1} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \hat{u_2} \\ \hat{v_2} \\ 1 \end{pmatrix}^\top \underbrace{\begin{pmatrix} 4f_{11} & 4f_{12} & 2f_{13} \\ 4f_{21} & 4f_{22} & 2f_{23} \\ 2f_{31} & 2f_{32} & f_{33} \end{pmatrix}}_{\hat{F}} \begin{pmatrix} \hat{u_1} \\ \hat{v_1} \\ 1 \end{pmatrix} .
\end{aligned}$$

8.2 Stereo Reconstruction

a) **Solution:** In order to reconstruct the depth, we need the inverses of the extrinsic and intrinsic matrices of both cameras. For the intrinsic matrices

$$A_1^{int} = A_2^{int} \quad = \quad \left(\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

the inveses can easily be computed as:

$$\left(A_1^{int}\right)^{-1} = \left(A_2^{int}\right)^{-1} \quad = \quad \left(\begin{array}{cc|c} 1 & 0 & -10 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) .$$

Now we have to compute the inverses for the extrinsic matrices

$$A_1^{ext} \quad = \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$A_2^{ext} \quad = \quad \left(\begin{array}{ccc|c} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -4 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) .$$

While the inverse of the first extrinsic matrix is the identity matrix, we use the following observation for rigid transformations in homogeneous coordinates to compute the inverse of the second extrinsic matrix:

$$A \quad = \quad \left(\begin{array}{c|c} R^{3\times 3} & t^3 \\ \hline 0^{3^\top} & 1 \end{array}\right)$$

$$A^{-1} \quad = \quad \left(\begin{array}{c|c} R^{-1} & R^{-1}t \\ \hline 0^{3^\top} & 1 \end{array}\right) .$$

Thus, the inverse of $(A_2^{ext})^{-1}$ is given by

$$\left(A_2^{ext}\right)^{-1} = \left(\begin{array}{ccc|c} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{4}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{4}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) .$$

b) **Solution:** Let us now recall how to go step by step from world coordinates to image coordinates.

$$\begin{array}{cl} \text{3-D world coordinates} & \\ \downarrow & \text{extrinsic parameters} \\ \text{3-D camera coordinates} & \\ \downarrow & \text{projection step} \\ \text{2-D image coordinates} & \\ \downarrow & \text{intrinsic parameters} \\ \text{2-D pixel coordinates} & \end{array}$$

If we want to reconstruct a point in 3-D space, we have to revert this order. In the first step we thus start with the the given 2-D Euclidean coordinates

$$\begin{aligned} x_1 &= (10.5, 0.5)^\top \\ x_2 &= \left(9.5, \frac{1}{\sqrt{2}}\right)^\top \end{aligned}$$

and compute the corresponding 2-D homogeneous coordinates that read

$$\begin{aligned} \hat{x_1} &= (10.5, 0.5, 1)^\top \\ \hat{x_2} &= \left(9.5, \frac{1}{\sqrt{2}}, 1\right)^\top . \end{aligned}$$

Using 2-D homogeneous coordinates as well as the inverses of the intrinsic matrices we can now determine the image coordinates from the pixel coordinates:

$$\begin{aligned} \left(A_1^{int}\right)^{-1} \hat{x_1} &= \begin{pmatrix} 1 & 0 & -10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 10.5 \\ 0.5 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \end{pmatrix} \\ \left(A_2^{int}\right)^{-1} \hat{x_2} &= \begin{pmatrix} 1 & 0 & -10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 9.5 \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -0.5 \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} . \end{aligned}$$

Let us now revert the projection process. To this end, we determine the 3-D Euclidean camera coordinates from the 2-D homogeneous image cooardinates:

$$X_1 = \lambda_1 \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \end{pmatrix} \quad , \lambda_1 \in \mathbb{R} \qquad \leftarrow \text{seen from first camera}$$

$$X_2 = \lambda_2 \begin{pmatrix} -0.5 \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} \quad , \lambda_2 \in \mathbb{R} \qquad \leftarrow \text{seen from the second camera}$$

As we can see, we obtain two rays in 3-D space.

Remark: Do not intersect these two rays! They live in different coordinate systems (the coordinate systems of camera 1 and 2, respectively).

c) **Solution:** Let us now transform these camera coordinates into world coordinates. To this end, we first have to transform the 3-D Euclidean coordinates into 3-D homogeneous coordinates:

$$\hat{X}_1 = (0.5\lambda_1, 0.5\lambda_1, \lambda_1, 1)^\top$$

$$\hat{X}_2 = \left(-0.5\lambda_2, \frac{1}{\sqrt{2}}\lambda_2, \lambda_2, 1\right)^\top$$

Now, we can compute the 3-D world coordinates of the two rays by multiplying their camera coordinates with the inverses of their corresponding extrinsic matrices:

$$\left(A_1^{ext}\right)^{-1} \hat{X}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5\lambda_1 \\ 0.5\lambda_1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5\lambda_1 \\ 0.5\lambda_1 \\ \lambda_1 \\ 1 \end{pmatrix}$$

$$\left(A_2^{ext}\right)^{-1}\hat{X}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{4}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{4}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -0.5\lambda_2 \\ \frac{1}{\sqrt{2}}\lambda_2 \\ \lambda_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2-\sqrt{2}}{4}\lambda_2 + 2\sqrt{2} \\ \frac{2+\sqrt{2}}{4}\lambda_2 - 2\sqrt{2} \\ \lambda_2 \\ 1 \end{pmatrix} .$$

Let us now determine the 3-D Euclidean coordinates of these rays from their 3-D homogeneous coordinates. Thus, we obtain

$$L_1 = \lambda_1 \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \end{pmatrix} \quad , \lambda_1 \in \mathbb{R}$$

$$L_2 = \lambda_2 \begin{pmatrix} \frac{2-\sqrt{2}}{4} \\ \frac{2+\sqrt{2}}{4} \\ 1 \end{pmatrix} + \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \\ 0 \end{pmatrix} \quad , \lambda_2 \in \mathbb{R} .$$

d) **Solution:** Now we are finally in the position to intersect these lines, since they are both given in 3-D Euclidean coordinates of the (same) world coordinate system:

$$\lambda_1 \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} \frac{2-\sqrt{2}}{4} \\ \frac{2+\sqrt{2}}{4} \\ 1 \end{pmatrix} + \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \\ 0 \end{pmatrix} \quad \leftarrow \text{ tells } \lambda_1 = \lambda_2$$

Substituting λ_2 with λ_1, we obtain

$$\lambda_1 \begin{pmatrix} \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} \\ 0 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \\ 0 \end{pmatrix}$$

from wich follows that

$$\lambda_1 = \lambda_2 = 8 .$$

The original 3-D point M has thus the world coordinates

$$M = (4, 4, 8)^\top$$

which results in a depth of $z = 8$.