

## 6.1 Warping and SOR

- a) **Solution:** Rearranging the two equations that we have to solve at each warping level  $k$  gives

$$\begin{aligned} J_{11}^k du^k + J_{12}^k dv^k - \alpha \Delta du^k &= \alpha \Delta u^k - J_{13}^k, \\ J_{12}^k du^k + J_{22}^k dv^k - \alpha \Delta dv^k &= \alpha \Delta v^k - J_{23}^k. \end{aligned}$$

Discretising this coupled system of partial differential equations, we then obtain

$$\begin{aligned} 0 &= [J_{11}^k]_{i,j} du_{i,j}^k + [J_{12}^k]_{i,j} dv_{i,j}^k + [J_{13}^k]_{i,j} - \alpha \sum \sum \frac{du_{\tilde{i},\tilde{j}}^k - du_{i,j}^k}{h_l^2}, \\ &\quad - \alpha \sum \sum \frac{u_{\tilde{i},\tilde{j}}^k - u_{i,j}^k}{h_l^2} \\ 0 &= [J_{12}^k]_{i,j} du_{i,j}^k + [J_{22}^k]_{i,j} dv_{i,j}^k + [J_{23}^k]_{i,j} - \alpha \sum \sum \frac{dv_{\tilde{i},\tilde{j}}^k - dv_{i,j}^k}{h_l^2}, \\ &\quad - \alpha \sum \sum \frac{v_{\tilde{i},\tilde{j}}^k - v_{i,j}^k}{h_l^2} \end{aligned}$$

where the double-summation has the following arguments (they are omitted due to limited space):

$$\sum_{l \in x,y} \sum_{(\tilde{i},\tilde{j}) \in \mathcal{N}_l(i,j)}.$$

From this we have to go to an equation system of the form  $A\mathbf{x} = \mathbf{b}$ . Let's define the following:

$$\mathbf{x} := \begin{pmatrix} du^k \\ dv^k \\ du^k \\ dv^k \end{pmatrix}, \quad \mathbf{b} := \begin{pmatrix} J_{13}^k \\ J_{13}^k \\ J_{23}^k \\ J_{23}^k \end{pmatrix} - \alpha \left( \begin{array}{ccc|ccc} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 1 & & & \\ \hline & & & 1 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{array} \right) \begin{pmatrix} u^k \\ v^k \\ u^k \\ v^k \end{pmatrix}$$

$$A := \left( \begin{array}{ccc|ccc} J_{11}^k & & & J_{12}^k & & \\ & J_{11}^k & & & J_{12}^k & \\ & & J_{11}^k & & & J_{12}^k \\ \hline J_{12}^k & & & J_{22}^k & & \\ & J_{12}^k & & & J_{22}^k & \\ & & J_{12}^k & & & J_{22}^k \end{array} \right) + \alpha \left( \begin{array}{ccc|ccc} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 1 & & & \\ \hline & & & 1 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{array} \right)$$

b) **Solution:** The structure of the double-summation is slightly changed here, since one has to distinguish between variables from the neighbourhood that have already been computed (denoted by  $\mathcal{N}_l^-$ ) and variables that have still to be computed and thus must be taken from the previous time step (denoted by  $\mathcal{N}_l^+$ ). First, let us set up the iteration step for the Gauß-Seidel method and then derive the iteration step for its SOR variant. The Gauß-Seidel step is given by:

$$du_{i,j}^{m+1,k} = \frac{\alpha \sum \sum \frac{u_{i,j}^k - u_{i,j}^k}{h_l^2} - [J_{13}^k]_{i,j} - \left( [J_{12}^k]_{i,j} dv_{i,j}^{m,k} - \alpha \sum \sum_{\mathcal{N}_l^-} \frac{du_{i,j}^{m+1,k}}{h_l^2} - \alpha \sum \sum_{\mathcal{N}_l^+} \frac{du_{i,j}^{m,k}}{h_l^2} \right)}{[J_{11}]_{i,j} + \alpha \sum \sum \frac{1}{h_l^2}}$$

$$dv_{i,j}^{m+1,k} = \frac{\alpha \sum \sum \frac{v_{i,j}^k - v_{i,j}^k}{h_l^2} - [J_{23}^k]_{i,j} - \left( [J_{12}^k]_{i,j} du_{i,j}^{m+1,k} - \alpha \sum \sum_{\mathcal{N}_l^-} \frac{dv_{i,j}^{m+1,k}}{h_l^2} - \alpha \sum \sum_{\mathcal{N}_l^+} \frac{dv_{i,j}^{m,k}}{h_l^2} \right)}{[J_{22}]_{i,j} + \alpha \sum \sum \frac{1}{h_l^2}}$$

where  $m$  is the iteration step and  $k$  the warping level.

**Remark:** In the formula for  $\tilde{d}v_{i,j}^{m+1,k}$  appears  $du_{i,j}$  with  $m+1, k$ . This is due to the fact that  $du$  is already computed, assuming the ordering  $du \rightarrow dv$  for the computation. Instead of using the results of the Gauß-Seidel method directly, the SOR method extrapolate its results:

$$du_{i,j}^{m+1,k} = (1 - \omega) du_{i,j}^{m,k} + \omega du_{i,j}^{(GS)m+1,k},$$

$$dv_{i,j}^{m+1,k} = (1 - \omega) dv_{i,j}^{m,k} + \omega dv_{i,j}^{(GS)m+1,k}.$$

Here,  $du_{i,j}^{(GS)m+1,k}$  and  $dv_{i,j}^{(GS)m+1,k}$  denote the result of the Gauß-Seidel method.

## 6.2 Photometric Invariants and Robustification

a) **Solution:** Let us first give an overview over the different properties where the following assumptions are considered:

- ① global multiplicative illumination changes
- ② shadow and shading (local multiplicative changes)
- ③ highlights and specular reflections (local additive / local multiplicative changes)

	①	②	③
$p_1$	<b>x</b>	<b>x</b>	<b>x</b>
$p_2$	<b>x</b>	<b>x</b>	<b>x</b>
$p_3$	✓	✓	<b>x</b>
$p_4$	✓	✓	✓
$p_5$	✓	<b>x</b>	<b>x</b>
$p_6$	✓	<b>x</b>	<b>x</b>

For the decision, what classes do apply, we consider Lecture 5, slide 21.

1. The first expressions is not invariant under any changing illumination due to

$$\begin{aligned}
 p_1 &= R - 3B + G \\
 &= e \left( m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x}) \right) - 3e \left( m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{B}_b(\mathbf{x}) \right) \\
 &\quad + e \left( m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{G}_b(\mathbf{x}) \right) \\
 &= e \left( -m_i(\mathbf{x})w_i + m_b(\mathbf{x})(\hat{R}_b(\mathbf{x}) - 3\hat{B}_b(\mathbf{x}) + \hat{G}_b(\mathbf{x})) \right)
 \end{aligned}$$

2. Here we have an invariance under local additive changes only.

$$\begin{aligned}
 p_2 &= R^2 + B^2 - 2BR = (R - B)^2 \\
 &= \left[ e \left( m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x}) - m_i(\mathbf{x})w_i - m_b(\mathbf{x})\hat{B}_b(\mathbf{x}) \right) \right]^2 \\
 &= \left[ em_b(\mathbf{x})(\hat{R}_b(\mathbf{x}) - \hat{B}_b(\mathbf{x})) \right]^2 \\
 &= e^2 m_b(\mathbf{x})^2 (\hat{R}_b(\mathbf{x}) - \hat{B}_b(\mathbf{x}))^2
 \end{aligned}$$

3. This expression is invariant under changes of the overall illumination.

$$\begin{aligned}
 p_3 &= \frac{R - B}{R + G} \\
 &= \frac{e \left( m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x}) - m_i(\mathbf{x})w_i - m_b(\mathbf{x})\hat{B}_b(\mathbf{x}) \right)}{e \left( m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x}) + m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{G}_b(\mathbf{x}) \right)} \\
 &= \frac{m_b(\mathbf{x}) \left( \hat{R}_b(\mathbf{x}) - \hat{B}_b(\mathbf{x}) \right)}{2m_i(\mathbf{x})w_i + m_b(\mathbf{x}) \left( \hat{R}_b(\mathbf{x}) + \hat{B}_b(\mathbf{x}) \right)} \\
 &= \frac{\hat{R}_b(\mathbf{x}) - \hat{B}_b(\mathbf{x})}{\hat{R}_b(\mathbf{x}) + \hat{B}_b(\mathbf{x})} \quad \text{if } m_i(\mathbf{x}) = 0.
 \end{aligned}$$

4. This one is invariant under local additive and multiplicative changes:

$$\begin{aligned}
 p_4 &= 2 \frac{BG + RG}{B^2 - R^2} - \frac{B + R}{B - R} \\
 &= 2 \frac{G(B + R)}{(B - R)(B + R)} - \frac{B + R}{B - R} = \frac{2G - B - R}{B - R} \\
 &= \frac{m_b(\mathbf{x}) \left( 2\hat{G}_b(\mathbf{x}) - \hat{B}_b(\mathbf{x}) - \hat{R}_b(\mathbf{x}) \right)}{m_b(\mathbf{x}) \left( \hat{B}_b(\mathbf{x}) - \hat{R}_b(\mathbf{x}) \right)} \\
 &= \frac{2\hat{G}_b(\mathbf{x}) - \hat{B}_b(\mathbf{x}) - \hat{R}_b(\mathbf{x})}{\hat{B}_b(\mathbf{x}) - \hat{R}_b(\mathbf{x})}
 \end{aligned}$$

5. Considering the following fact that

$$\ln(\underbrace{c}_{\text{factor}} B)_x = (\underbrace{\ln c}_{\text{constant}} + \ln B)_x = (\ln B)_x$$

this expressions is invariant under global multiplicative changes:

$$\begin{aligned}
 p_5 &= (\ln B)_x + (\ln R)_x \\
 &= \frac{(m_{ix}(\mathbf{x})w_i + m_{bx}(\mathbf{x})\hat{B}_b(\mathbf{x}) + m_b(\mathbf{x})\hat{B}_{bx}(\mathbf{x}))}{\left[ (m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{B}_b(\mathbf{x})) \right]} \\
 &\quad + \frac{(m_{ix}(\mathbf{x})w_i + m_{bx}(\mathbf{x})\hat{R}_b(\mathbf{x}) + m_b(\mathbf{x})\hat{R}_{bx}(\mathbf{x}))}{\left[ (m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x})) \right]}.
 \end{aligned}$$

6. As given in the exercise, this expression is invariant under local additive and global multiplicative changes

$$\begin{aligned} p_6 &= \ln B_x - \ln R_x = \ln(B_x/R_x) \\ &= \ln \left[ \frac{\left( (m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{B}_b(\mathbf{x})) \right)_x}{\left( (m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x})) \right)_x} \right] \end{aligned}$$

As discussed in the tutorials (here we used a plus instead of the minus), it is invariant under global additive changes only:

$$\begin{aligned} p_6 &= \ln B_x + \ln R_x = \ln(B_x \cdot R_x) \\ &= \ln \left[ \left( e(m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{B}_b(\mathbf{x})) \right)_x \cdot \left( e(m_i(\mathbf{x})w_i + m_b(\mathbf{x})\hat{R}_b(\mathbf{x})) \right)_x \right] \end{aligned}$$

- b) **Solution:** Since we have to integrate six constant assumptions, we will have a sum over six channels in the data term. Additionally, we are not allowed to linearise these assumptions, since this would not work for large displacements. Robustness can be achieved by robustification of the data term. Here we have two choices: Joint and separate robustification. Both cases will be discussed in the following. To keep things simple we decide to use the quadratic smoothness term of Horn and Schunck. Let us now write down both energy functionals for joint and separate robustification. To this end, we define the robust function  $\Psi(s^2)$  as follows:

$$\Psi(s^2) = \sqrt{s^2 + \varepsilon^2} \quad \text{with } \varepsilon = 10^{-3}.$$

The energy functional for the joint robustification is then given by:

$$\begin{aligned} E_{joint}(u, v) &= \int_{\Omega} \Psi \left( \sum_{i=1}^6 (p_i(x+u, y+v, t+1) - p_i(x, y, t))^2 \right) \\ &\quad + \alpha (|\nabla u|^2 + |\nabla v|^2) \, dx dy . \end{aligned}$$

The corresponding variant with separate robustification reads:

$$\begin{aligned} E_{sep}(u, v) &= \int_{\Omega} \sum_{i=1}^6 \Psi \left( (p_i(x+u, y+v, t+1) - p_i(x, y, t))^2 \right) \\ &\quad + \alpha (|\nabla u|^2 + |\nabla v|^2) \, dx dy . \end{aligned}$$

c) **Solution:** Let us now calculate the Euler-Lagrange equations. Their general form is given by

$$\begin{aligned} 0 &= F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} \\ 0 &= F_v - \frac{\partial}{\partial x} F_{v_x} - \frac{\partial}{\partial y} F_{v_y} \end{aligned}$$

• joint robustification:

$$\begin{aligned} F &= \Psi \left( \sum_{i=1}^6 (p_i(x+u, y+v, t+1) - p_i(x, y, t))^2 \right) \\ &\quad + \alpha (|\nabla u|^2 + |\nabla v|^2) \\ F_u &= 2 \cdot \Psi'(\dots) \\ &\quad \cdot \sum_{i=1}^6 (p_i(x+u, y+v, t+1) - p_i(x, y, t)) \\ &\quad \cdot p_{i_x}(x+u, y+v, t+1) \\ F_v &= 2 \cdot \Psi'(\dots) \\ &\quad \cdot \sum_{i=1}^6 (p_i(x+u, y+v, t+1) - p_i(x, y, t)) \\ &\quad \cdot p_{i_y}(x+u, y+v, t+1) \\ F_{u_x} &= 2\alpha u_x \\ F_{u_y} &= 2\alpha u_y \\ F_{v_x} &= 2\alpha v_x \\ F_{v_y} &= 2\alpha v_y \end{aligned}$$

The Euler-Lagrange equations now read as (divided by 2):

$$\begin{aligned} 0 &= \Psi'(\sum \dots) \left( \sum_{i=1}^6 (p_i(x+u, y+v, t+1) - p_i(x, y, t)) p_{i_x}(x+u, y+v, t+1) \right) - \alpha \Delta u \\ 0 &= \Psi'(\sum \dots) \left( \sum_{i=1}^6 (p_i(x+u, y+v, t+1) - p_i(x, y, t)) p_{i_y}(x+u, y+v, t+1) \right) - \alpha \Delta v \end{aligned}$$

As one can see, there is only a joint weight  $\Psi'(\sum \dots)$  for all constancy assumptions. If one assumption is violated, the argument of this expressions is large. Knowing that

$$\Psi'(s^2) = \frac{1}{2\sqrt{s^2 + \varepsilon^2}}$$

becomes small for large arguments, we can notice that in such cases all constancy assumptions are turned of and the smoothness term takes over. Unfortunately, this make only sense, if all assumptions are correlated (dependent). Then one can deduce from the failure of constancy assumptions that the other ones are violated as well. In the case of uncorrlated (independent) assumptions, using the joint model may ignore important information from those assumptions that are not violated.

- separate robustification:

$$\begin{aligned}
 F &= \sum_{i=1}^6 \Psi \left( (p_i(x+u, y+v, t+1) - p_i(x, y, t))^2 \right) \\
 &\quad + \alpha (|\nabla u|^2 + |\nabla v|^2) \\
 F_u &= 2 \cdot \sum_{i=1}^6 \Psi'(\dots) \cdot (p_i(x+u, y+v, t+1) - p_i(x, y, t)) \\
 &\quad \cdot p_{i_x}(x+u, y+v, t+1) \\
 F_v &= 2 \cdot \sum_{i=1}^6 \Psi'(\dots) \cdot (p_i(x+u, y+v, t+1) - p_i(x, y, t)) \\
 &\quad \cdot p_{i_y}(x+u, y+v, t+1) \\
 F_{u_x} &= 2\alpha u_x \\
 F_{u_y} &= 2\alpha u_y \\
 F_{v_x} &= 2\alpha v_x \\
 F_{v_y} &= 2\alpha v_y
 \end{aligned}$$

The Euler-Lagrange equations read as (again divided by 2):

$$\begin{aligned}
 0 &= \sum_{i=1}^6 \left( \Psi'(\dots)(p_i(x+u, y+v, t+1) - p_i(x, y, t)) \cdot p_{i_x}(x+u, y+v, t+1) \right) - \alpha \Delta u \\
 0 &= \sum_{i=1}^6 \left( \Psi'(\dots)(p_i(x+u, y+v, t+1) - p_i(x, y, t)) \cdot p_{i_y}(x+u, y+v, t+1) \right) - \alpha \Delta v
 \end{aligned}$$

In this case, we have a separate weight  $\Psi'(\dots)$  for each constancy assumption. If one assumption is violated only that assumption is weighted down. If the other assumptions are fulfilled, they take over and compute the flow together with the smoothness term.