4.1 Affine Horn and Schunck

a) **Solution:** We know from the lecture that the affine parameterisation of the optic flow is given by:

$$\mathbf{w} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} ax+by+c \\ dx+ey+f \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} x & y & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ 1 \end{pmatrix}}_{\mathbf{p}} = M\mathbf{p}$$

Thus the squared data term for the grey value constancy assumption with affine parametrisation can be written as:

$$(\nabla_3 f^\top \mathbf{w})^2 = (\nabla_3 f^\top (M\mathbf{p}))^2 = \mathbf{p}^\top \underbrace{\underbrace{M^\top \nabla_3 f}_{r} \underbrace{\nabla_3 f^\top M}_{r^\top}}_{J} \mathbf{p}$$

where the corresponding motion tensor J can be computed via the tensor product from the vector

$$r = \begin{pmatrix} xf_x \\ yf_x \\ f_x \\ xf_y \\ yf_y \\ f_y \\ f_t \end{pmatrix} .$$

While we can substitute the original data term of Horn and Schunck by its parametrised version, we have to modify the smoothness term in order to obtain different results than the original method. To this end, we replace the original regularizer that assumes the flow field u, v to by a smoothness term that assumes the affine parameters to be smooth (i.e. that the flow field is affine). In this way, we obtain the new energy functional:

$$E(\mathbf{p}) = \int_\Omega \mathbf{p}^\top J \mathbf{p} + \alpha(|\nabla a|^2 + |\nabla b|^2 + |\nabla c|^2 + |\nabla d|^2 + |\nabla e|^2 + |\nabla f|^2)\, \mathrm{d}x\mathrm{d}y.$$

Considering the fact that the parameters a, b, d, and e describe slopes while the parameters c and d describe offsets, one may penalise them using differnt smoothness variables: Another possibility is

$$E(\mathbf{p}) = \int_\Omega \mathbf{p}^\top J\mathbf{p} + \alpha(|\nabla a|^2+|\nabla b|^2+|\nabla d|^2+|\nabla e|^2) + \beta(|\nabla c|^2+|\nabla f|^2)\, \mathrm{d}x\mathrm{d}y.$$

b) **Solution:** For the computation of the Euler-Langrange equations, we stick to the first proposal for the energy functional. They are given by

$$\begin{aligned} 0 &= J_{11}a + J_{12}b + J_{13}c + J_{14}d + J_{15}e + J_{16}f + J_{17} - \alpha\Delta a \\ 0 &= J_{12}a + J_{22}b + J_{23}c + J_{24}d + J_{25}e + J_{26}f + J_{27} - \alpha\Delta b \\ &\vdots \\ 0 &= J_{61}a + J_{62}b + J_{63}c + J_{64}d + J_{65}e + J_{66}f + J_{67} - \alpha\Delta f \end{aligned}$$

Since we have 6 affine parameters, it is not surprising that we obtain 6 Euler-Lagrange equations in total.

4.2 Motion Tensors

a) **Solution:** Assuming the determinant of the Hessian to be constant, we obtain the constraint

$$\det(\mathcal{H}_2 f(x,y,t)) - \det(\mathcal{H}_2 f(x+u,y+v,t+1)) = 0\ .$$

By evaluating the Hessians this constraint can be formulated as:

$$\begin{aligned} &\Big(f_{xx}(x,y,t)\cdot f_{yy}(x,y,t) - f_{xy}^2(x,y,t)\Big) \\ -\ &\Big(f_{xx}(x{+}u,y{+}v,t{+}1)\cdot f_{yy}(x{+}u,y{+}v,t{+}1) - f_{xy}^2(x{+}u,y{+}v,t{+}1)\Big) = 0\ . \end{aligned}$$

Linearisation then gives

$$\Big(\underbrace{f_{xx}f_{yy} - f_{xy}^2}_{=:p}\Big)_x u + \Big(\underbrace{f_{xx}f_{yy} - f_{xy}^2}_{=:p}\Big)_y v + \Big(\underbrace{f_{xx}f_{yy} - f_{xy}^2}_{=:p}\Big)_t = 0\ ,$$

where p is the determinant of the Hessian. The required partial derivatives of p can then be computed as

$$\begin{aligned} p_x &= f_{xxx}\cdot f_{yy} + f_{xx}\cdot f_{xyy} - 2f_{xy}f_{xxy} \\ p_y &= f_{xxy}\cdot f_{yy} + f_{xx}\cdot f_{yyy} - 2f_{xy}f_{xyy} \\ p_t &= f_{xxt}\cdot f_{yy} + f_{xx}\cdot f_{yyt} - 2f_{xy}f_{xyt} \\ \Leftrightarrow \nabla p &= \nabla_3 f_{xx}\cdot f_{yy} + \nabla_3 f_{yy}\cdot f_{xx} - 2f_{xy}\nabla_3 f_{xy} \end{aligned}$$

The corresponding motion is now given as the tensor product of ∇p:

$$J = \nabla p \nabla_3 p^\top \ .$$

Since there is only one equation and two variables, the aperture problem is always present

b) **Solution:** Extending the motion tensor to RGB colour images is straighforward. One only has to redefine p in the following way: We now use $(f_1, f_2, f_3)^\top = (R, G, B)^\top$ as our function f, giving us one equation for every channel. We now have three equations

$$\Big(\underbrace{f_{i_{xx}} f_{i_{yy}} - f_{i_{xy}}^2}_{=:p}\Big)_x u + \Big(\underbrace{f_{i_{xx}} f_{i_{yy}} - f_{i_{xy}}^2}_{=:p}\Big)_y v + \Big(\underbrace{f_{i_{xx}} f_{i_{yy}} - f_{i_{xy}}^2}_{=:p}\Big)_t = 0$$

for $i = 1, 2, 3$. The corresponding motion tensor is then given by

$$J = \sum_{i=1}^{3} \nabla p_i \nabla_3 p_i^\top \ .$$

Since we have now three constraints for two unknown, there may be situation where the data term is sufficient to determine the flow uniquely. However, if not sufficient local variations take place in all three channels (e.g. in homogeneous areas), the aperture problem can still be present.