2.1 Sub-Pixel Refinement I

Solution: Fitting the parabola $f(x) = ax^2 + bx + x$ to the cost values $f(0) = c_0$, $f(-1) = c_{-1}$, and $f(1) = c_{+1}$ yields the three equations

$$\begin{aligned} f(0) &= a \cdot 0^2 + b \cdot 0 + c \,, && (1) \\ f(-1) &= a \cdot 1 + b \cdot (-1) + c \,, && (2) \\ f(1) &= a \cdot 1 + b \cdot 1 + c \,. && (3) \end{aligned}$$

From (1) we know that $c_0 = c$. From (2), we get

$$c_{-1} = a - b + c_0$$

and from (3)

$$c_{+1} = a + b + c_0.$$

Solving these two latter equations for a and combining them to eliminate this variable, we can compute b:

$$\begin{aligned} c_{-1} + b - c_0 &= c_{+1} - b - c_0 \\ \Leftrightarrow 2b &= c_{+1} - c - -1 \\ \Leftrightarrow b &= \frac{c_{+1} - c_{-1}}{2} \end{aligned}$$

This can now be used to compute a:

$$\begin{aligned} c_{+1} &= a + b + c_0 \\ \Leftrightarrow a &= c_{+1} - b - c_0 \\ &= c_{+1} - \frac{c_{+1} - c_{-1}}{2} - c_0 \\ a &= \frac{c_{+1} - 2c_0 - c_{-1}}{2} \end{aligned}$$

□

2.2 Sub-Pixel Refinement II

Remark: There is a small error in the given result for the case $c_{+1} < c_{-1}$ where the stated result for the variable b has the wrong sign.
Solution: As in the previous case, the given points give us three equations:

$$\begin{aligned} f(-1) &= a \cdot |-1 + b| + c = c_{-1} \\ f(0) &= a \cdot |b| + c = c_0 \\ f(1) &= a \cdot |1 + b| + c = c_{+1} \end{aligned}$$

Unfortunately, these equations are now nonlinear due to the sign function. However, we know from the hint that $-1 \leq b \leq 1$. So we have to consider only the cases $b < 0$ and $b \geq 0$:

- $b < 0$:

$$\begin{aligned} -ab + a + c &= c_{-1} && (4) \\ -ab + c &= c_0 && (5) \\ ab + a + c &= c_{+1} && (6) \end{aligned}$$

Now we can substract (4) and (5), resulting in

$$a = c_{-1} - c_0.$$

Adding up (4) and (5) results in

$$\begin{aligned} 2a + 2c &= c_{+1} + c_{-1} \\ 2(c_{-1} - c_0) + 2c &= c_{+1} - c_{-1} \\ 2c &= c_{+1} + c_{-1} - 2c_{-1} + 2c_0 \\ c &= \frac{c_{+1} + 2c_0 - c_{-1}}{2} \end{aligned}$$

Finally, we can calculate b as

$$b = \frac{c_{+1} - c_{-1}}{2(c_{-1} - c_0)}.$$

- $b \geq 0$:

$$\begin{aligned} -ab + a + c &= c_{-1} && (7) \\ ab + c &= c_0 && (8) \\ ab + a + c &= c_{+1} && (9) \end{aligned}$$

If we substract (8) from (9), we get

$$a = c_{+1} - c_0.$$

Adding up (7) and (9) gives

$$\begin{aligned} 2a + 2c &= c_{+1} + c_{-1} \\ 2(c_{+1} - c_0) + 2c &= c_{+1} + c_{-1} \\ 2c &= c_{+1} + c_{-1} - 2c_{+1} + 2c_0 \\ c &= \frac{c_{-1} + 2c_0 - c_{+1}}{2} \end{aligned}$$

Now we can calculate the (correct!) b:

$$\begin{aligned}
ab + c &= c_0 \\
b &= \frac{c_0 - c}{a} \\
&= \frac{c_0 - \frac{c_{-1} + 2c_0 - c_{+1}}{2}}{c_{+1} - c_0} \\
b &= \frac{c_{+1} - c_{-1}}{2(c_{+1} - c_0)}
\end{aligned}$$

□