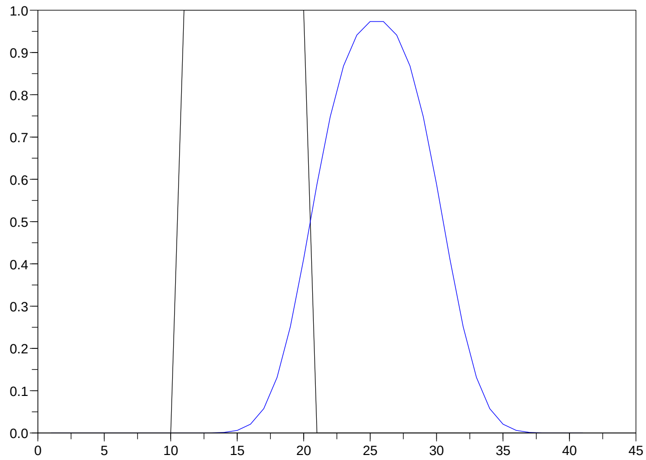
Example Solutions for Assignment 7

Problem 1 (Shake That!)

1. We will be using a discretised signal on the interval $[-1, 3]$ with the discretised points having a distance of $\Delta x = 0.1$ between each other, together with the given definition for the signal by equation (3). This gives a standard box function. Concerning boundary conditions, we will use periodic boundary conditions, i.e. $u_0 = u_n$ and $u_{n+1} = u_1$ for the signal with length n. With this setup, the signal propagates towards the right direction, however is then being repeated at the left hand side of the interval. Standard von Neumann boundaries would have the effect of washing the entire signal towards one boundary.



2. **General remarks:**

   - Viscosity solutions satisfy a minimum-maximum-principle enforced by the diffusion term εu_{xx}, see (13.12).
   - Monotone schemes satisfy a discrete minimum-maximum-principle.

   Question: Can we recognise a similar form as the viscosity solution provides by analysing the numerical scheme? As an example, we consider the upwind scheme

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) \qquad (1)$$

approximating $u_t + au_x = 0, \quad a > 0$. Taylor expansions yield, employing $u \equiv u(j\Delta x, n\Delta t) \approx u_j^n$:

$$u_j^{n+1} = u + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + \mathcal{O}(\Delta t^4) \tag{2a}$$

$$u_j^n = u \tag{2b}$$

$$u_{j-1}^n = u - \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} - \frac{\Delta x^3}{6} u_{xxx} + \mathcal{O}(\Delta x^4) \tag{2c}$$

Computing the local truncation error L, we obtain by use of (2a) and $\lambda = \frac{\Delta t}{\Delta x} =$ constant:

$$\begin{aligned} L &= \frac{[u + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + \mathcal{O}(\Delta t^4)] - u}{\Delta t} \\ &\quad + \frac{a}{\Delta x}\left[u - \left[u - \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} - \frac{\Delta x^3}{6} u_{xxx} + \mathcal{O}(\Delta x^4)\right]\right] \\ &= \left.\begin{array}{l} u_t + \frac{\Delta t}{2} u_{tt} + \frac{\Delta t^2}{6} u_{ttt} + \mathcal{O}(\Delta t^3) \\ + a u_x - \frac{a\Delta x}{2} u_{xx} + \frac{a \Delta x^2}{6} u_{xxx} + \mathcal{O}(\Delta x^3) \end{array}\right\} \quad (3a) \\ &= \frac{\Delta t}{2} u_{tt} + \frac{\Delta t^2}{6} u_{ttt} - \frac{a\Delta x}{2} u_{xx} + \frac{a\Delta x^2}{6} u_{xxx} + \mathcal{O}(\Delta x^3). \quad (3b) \end{aligned}$$

We observe, that the upwind scheme is of first order in space and time.

3. In order to give (3a) a further interpretation, we use the approximated PDE $u_t + au_x = 0$ again:

$$\begin{aligned} u_{tt} = (u_t)_t &= (-au_x)_t \\ &= -a(u_t)_x \\ &= -a(-au_x)_x = a^2 u_{xx}. \end{aligned} \tag{4}$$

4. Plugging this into (3a) gives

$$\begin{aligned} L &= \frac{\Delta t}{2} a^2 u_{xx} - \frac{a\Delta x}{2} u_{xx} + \mathcal{O}(\Delta x^2) \\ &= -\frac{a\Delta x}{2}(1 - a\lambda) u_{xx} + \mathcal{O}(\Delta x^2). \end{aligned} \tag{5}$$

Thus, $-\frac{a\Delta x}{2}(1 - a\lambda)u_{xx}$ is the <u>leading order error term</u>. The <u>idea</u> is now to subtract the leading order error term from the original equation, thus obtaining a PDE-model for the qualitative behaviour of the scheme.

The statement is here: The upwind scheme

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) \tag{6}$$

is a first-order accurate approximation of the PDE

$$u_t + au_x = 0, \tag{7}$$

and a second-order accurate approximation of the modified equation

$$u_t + au_x = \frac{a\Delta x}{2}(1 - a\lambda)u_{xx}. \tag{8}$$

5. **Remarks:**
   - For $1 - a\lambda \geq 0 \Leftrightarrow \Delta t \leq \frac{\Delta x}{a}$, (8) is of the same form as the advection-diffusion PDE (13.12). Here we observe the link to viscosity solutions, suggesting that the scheme is reasonable.
   - The condition $\Delta t \leq \frac{\Delta x}{a}$ is exactly the CFL-condition in the monotonicity analysis. For larger Δt, (8) suggests the influence of backward diffusion, leading to a blow-up of numerical solutions.

To verify the link between (6) and (8), we have to use the same procedure as for the computation of the local truncation error, but with the exception of the trick (4) as now the underlying PDE is (8) and not (7):

$$\begin{aligned}
u_{tt} = (u_t)_t &= (-au_x + \frac{a\Delta x}{2}(1 - a\lambda)u_{xx})_t \\
&= -a(u_t)_x + \mathcal{O}(\Delta x) \\
&= -a(-au_x + \frac{a\Delta x}{2}(1 - a\lambda)u_{xx})_x + \mathcal{O}(\Delta x) \\
&= a^2 u_{xx} + \mathcal{O}(\Delta x).
\end{aligned} \tag{9}$$

Plugging (9) and the PDE (8) into the computation of L yields

$$\begin{aligned}
L &\overset{(3a)}{=} u_t + \frac{\Delta t}{2}u_{tt} + \frac{\Delta t^2}{6}u_{ttt} \\
&\quad +au_x - \frac{a\Delta x}{2}u_{xx} + \frac{a\Delta x^2}{6}u_{xxx} + \mathcal{O}(\Delta x^3) \\
&= \frac{a\Delta x}{2}(1 - a\lambda)u_{xx} + \frac{\Delta t}{2}(a^2 u_{xx} + \mathcal{O}(\Delta x)) - \frac{a\Delta x}{2}u_{xx} + \mathcal{O}(\Delta x^2) \\
&= \frac{a\Delta x}{2}(1 - a\lambda)u_{xx} + a^2\frac{\Delta t}{2}u_{xx} - \frac{a\Delta x}{2}u_{xx} + \mathcal{O}(\Delta x^2) - \frac{a\Delta x}{2}(1 - a\lambda)u_{xx} \\
&= \mathcal{O}(\Delta x^2).
\end{aligned}$$

Problem 2 (No Time to Chill)

1. By use of CFL-condition (14.10), it must hold that

$$\begin{aligned} \frac{\partial H}{\partial u_{j-1}^n} &= \frac{a\Delta t}{\Delta x} \geq 0 \qquad \text{for } a > 0, \\ \frac{\partial H}{\partial u_j^n} &= 1 - \frac{a\Delta t}{\Delta x} \overset{!}{\geq} 0 \qquad \Leftrightarrow \Delta t \leq \frac{\Delta x}{a} \end{aligned}$$

   The first condition is met trivially, whereas the second condition is being computed directly by inserting the given parameters. Thus, we achieve $\Delta t \leq 0.4$ for $a_1 = 0.5$ and $\Delta t \leq 0.04$ for $a_2 = 5$.

2. The CFL-conditions imply monotony of the scheme. Monotony of the scheme itself implies stability. Thus, the derived CFL-conditions are valid stability conditions.