Numerical Algorithms for Visual Computing 2008/09
Example Solutions for Assignment 4

Problem 1 (Shift invariance of the Laplace operator)

The Laplace equation is shift invariant, i.e., invariant under translations

$$x' = x + a \quad , \quad y' = y + b, \qquad a, b \in \mathbb{R}.$$

The shift invariance can then be written as

$$u_{xx} + u_{yy} \quad = \quad u_{x'x'} + u_{y'y'}$$

To see this explicitly, we consider x, y as mappings depending on x' and y', respectively, and we compute

$$y(y') = y' - b, \quad x(x') = x' - a \quad \Rightarrow \quad \frac{\partial y(y')}{\partial y'} = 1, \frac{\partial x(x')}{\partial x'} = 1.$$

It is not wrong to consider $x \equiv x(x', y'), y \equiv y(x', y')$, so that

$$\frac{\partial y}{\partial y'} = 1 \quad , \quad \frac{\partial x}{\partial x'} = 1.$$

It follows

$$\begin{aligned}
u_{x'x'} = u(x,y)_{x'x'} &= \frac{\partial}{\partial x'}\left(\frac{\partial}{\partial x'} u(x,y)\right) \\
&= \frac{\partial}{\partial x'}\left(\underbrace{\frac{\partial x}{\partial x'}}_{=1} \cdot \frac{\partial}{\partial x} u(x,y)\right) \\
&= \frac{\partial}{\partial x'} u_x(x,y) \\
&= \frac{\partial}{\partial x} \underbrace{\frac{\partial x}{\partial x'}}_{=1} u_x(x,y) = u_{xx}(x,y).
\end{aligned}$$

$u_{y'y'} = u_{yy}$ follows analogously.

Problem 2 (What's the matrix, what's the matrix?)

1. For the ordering

$$(u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \quad u_7 \quad u_8 \quad u_9 \quad u_{10} \quad u_{11} \quad u_{12} \quad u_{13} \quad u_{14} \quad u_{15} \quad u_{16})^\top \qquad (1)$$

and the underlying process

$$-\left[\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{\Delta y^2}\right] \quad = \quad f_{ij} \qquad (2)$$

we get the following matrix system

$$\left|\begin{array}{cccc|cccc|cccc|cccc}
\hline
4 & -1 & & & -1 & & & & & & & & & & & \\
-1 & 4 & -1 & & & -1 & & & & & & & & & & \\
 & -1 & 4 & -1 & & & -1 & & & & & & & & & \\
 & & -1 & 4 & & & & -1 & & & & & & & & \\
\hline
-1 & & & & 4 & -1 & & & -1 & & & & & & & \\
 & -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
 & & -1 & & & -1 & 4 & -1 & & & -1 & & & & & \\
 & & & -1 & & & -1 & 4 & & & & -1 & & & & \\
\hline
 & & & & -1 & & & & 4 & -1 & & & -1 & & & \\
 & & & & & -1 & & & -1 & 4 & -1 & & & -1 & & \\
 & & & & & & -1 & & & -1 & 4 & -1 & & & -1 & \\
 & & & & & & & -1 & & & -1 & 4 & & & & -1 \\
\hline
 & & & & & & & & -1 & & & & 4 & -1 & & \\
 & & & & & & & & & -1 & & & -1 & 4 & -1 & \\
 & & & & & & & & & & -1 & & & -1 & 4 & -1 \\
 & & & & & & & & & & & -1 & & & -1 & 4 \\
\hline
\end{array}\right| \qquad (3)$$

2. For the ordering

$$(u_1 \quad u_3 \quad u_6 \quad u_{10} \quad u_2 \quad u_5 \quad u_9 \quad u_{13} \quad u_4 \quad u_8 \quad u_{12} \quad u_{15} \quad u_7 \quad u_{11} \quad u_{14} \quad u_{16})^\top \qquad (4)$$

we get the following matrix system

$$
\left[\begin{array}{c|cc|ccc|cccc|ccc|cc|c}
\hline
4 & -1 & -1 & & & & & & & & & & & & & \\
\hline
-1 & 4 & & -1 & -1 & & & & & & & & & & & \\
-1 & & 4 & & -1 & -1 & & & & & & & & & & \\
\hline
 & -1 & & 4 & & & -1 & -1 & & & & & & & & \\
 & -1 & -1 & & 4 & & & -1 & -1 & & & & & & & \\
 & & -1 & & & 4 & & & -1 & -1 & & & & & & \\
\hline
 & & & -1 & & & 4 & & & & -1 & & & & & \\
 & & & -1 & -1 & & & 4 & & & -1 & -1 & & & & \\
 & & & & -1 & -1 & & & 4 & & & -1 & -1 & & & \\
 & & & & & -1 & & & & 4 & & & -1 & & & \\
\hline
 & & & & & & -1 & -1 & & & 4 & & & -1 & & \\
 & & & & & & & -1 & -1 & & & 4 & & -1 & -1 & \\
 & & & & & & & & -1 & -1 & & & 4 & & -1 & \\
\hline
 & & & & & & & & & & -1 & -1 & & 4 & & -1 \\
 & & & & & & & & & & & -1 & -1 & & 4 & -1 \\
\hline
 & & & & & & & & & & & & & -1 & -1 & 4 \\
\hline
\end{array}\right] \tag{5}
$$

Problem 3 (Crossing derivatives)

1. We have been given the following cross derivative discretisation by use of central difference methods

$$
\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial}{\partial y} &= \frac{\partial}{\partial x}\left(\frac{u_{i,j+1} - u_{i,j-1}}{2h}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{u_{i,j+1}}{2h}\right) - \frac{\partial}{\partial x}\left(\frac{u_{i,j-1}}{2h}\right) \\
&= \frac{u_{i+1,j+1} - u_{i-1,j+1}}{4h^2} - \frac{u_{i+1,j-1} - u_{i-1,j-1}}{4h^2} \\
&= \frac{1}{4h^2}(u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1})
\end{aligned}
$$

with $h = \Delta x = \Delta y$. For this 4 points we can compute the 2D-

Taylorexpansion

$$\begin{aligned} u(x_0, y_0) &= f(x,y) + \binom{1}{0}\frac{\partial}{\partial x} f(x,y)(x-x_0) + \binom{1}{1}\frac{\partial}{\partial y} f(x,y)(y-y_0) \\ &+\frac{1}{2}\left(\binom{2}{0}\frac{\partial^2}{\partial x^2} f(x,y)(x-x_0)^2 + \binom{2}{1}\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y)(x-x_0)(y-y_0)\right. \\ &\left. + \binom{2}{2} f(x,y)\frac{\partial^2}{\partial y^2}(y-y_0)^2\right) \end{aligned}$$

This gives for our simple points the following approximation:

$$\begin{aligned} u(x \pm h, y+h) = u &\pm hu_x + u_y + \frac{1}{2}h^2(u_{xx} \pm 2u_{xy} + u_{yy}) \\ &+ \frac{1}{6}h^3(\pm u_{xxx} + 3u_{xxy} \pm u_{xyy} + 3u_{yyy}) \\ &+ \frac{1}{24}h^4(u_{xxxx} \pm 4u_{xxxy} + 6u_{xxyy} \pm 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5) \\ u(x+h, y-h) = u &+ hu_x - u_y + \frac{1}{2}h^2(u_{xx} - 2u_{xy} + u_{yy}) \\ &+ \frac{1}{6}h^3(u_{xxx} - 3u_{xxy} + u_{xyy} - 3u_{yyy}) \\ &+ \frac{1}{24}h^4(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5) \\ u(x-h, y-h) = u &- hu_x - u_y + \frac{1}{2}h^2(u_{xx} + 2u_{xy} + u_{yy}) \\ &- \frac{1}{6}h^3(u_{xxx} + 3u_{xxy} + u_{xyy} + 3u_{yyy}) \\ &+ \frac{1}{24}h^4(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5) \end{aligned}$$

Now we can input this approximation into our four pixel scheme:

$$
\begin{aligned}
u_{xy} &\approx \frac{1}{4h^2}(u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}) \\
&= \frac{1}{4h^2}(u + hu_x + u_y + \frac{1}{2}h^2(u_{xx} + 2u_{xy} + u_{yy}) \\
&+ \frac{1}{6}h^3(u_{xxx} + 3u_{xxy} + u_{xyy} + 3u_{yyy}) \\
&+ \frac{1}{24}h^4(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5) \\
&\quad -(u - hu_x + u_y + \frac{1}{2}h^2(u_{xx} - 2u_{xy} + u_{yy}) \\
&+ \frac{1}{6}h^3(-u_{xxx} + 3u_{xxy} - u_{xyy} + 3u_{yyy}) \\
&+ \frac{1}{24}h^4(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5)) \\
&\quad -(u + hu_x - u_y + \frac{1}{2}h^2(u_{xx} - 2u_{xy} + u_{yy}) \\
&+ \frac{1}{6}h^3(u_{xxx} - 3u_{xxy} + u_{xyy} - 3u_{yyy}) \\
&+ \frac{1}{24}h^4(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5)) \\
&\quad +u - hu_x - u_y + \frac{1}{2}h^2(u_{xx} + 2u_{xy} + u_{yy}) \\
&- \frac{1}{6}h^3(u_{xxx} + 3u_{xxy} + u_{xyy} + 3u_{yyy}) \\
&+ \frac{1}{24}h^4(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5))
\end{aligned}
$$

If we combine this terms together, we will get:

$$
\begin{aligned}
\frac{1}{4h^2} \quad & (u\underbrace{(1-1-1+1)}_{=0} + hu_x\underbrace{(1+1-1-1)}_{=0} + hu_y\underbrace{(1-1+1-1)}_{=0} \\
+ \quad & \frac{1}{2}h^2\underbrace{(u_{xx}+2u_{xy}+u_{yy}-u_{xx}+2u_{xy}-u_{yy}-u_{xx}+2u_{xy}-u_{yy}+u_{xx}+2u_{xy}+u_{yy})}_{8u_{xy}} \\
+ \quad & \frac{1}{6}h^3(u_{xxx}\underbrace{(1+1-1-1)}_{=0} + 3u_{xxy}\underbrace{(1-1+1-1)}_{=0} \\
+ \quad & 3u_{xyy}\underbrace{(1+1-1-1)}_{=0} + u_{yyy}\underbrace{(1-1+1-1)}_{=0}) \\
+ \quad & \frac{1}{24}h^4(u_{xxxx}\underbrace{(1-1-1+1)}_{=0} + 4u_{xxxy}\underbrace{(1+1+1+1)}_{=4} \\
& +6u_{xxyy}\underbrace{(1-1-1+1)}_{=0} + 4u_{xyyy}\underbrace{(1+1+1+1)}_{=4} + u_{yyyy}\underbrace{(1-1-1+1)}_{=0}) + \mathcal{O}(h^5))
\end{aligned}
$$

This sums up to

$$
\begin{aligned}
& \frac{1}{4h^2}(4h^2u_{xy} + \frac{16}{24}(u_{xxxy}+u_{xyyy}) + \mathcal{O}(h^5)) \\
= \quad & u_{xy} + \frac{1}{6}h^2(u_{xxxy}+u_{xyyy}) + \mathcal{O}(h^3) \\
= \quad & u_{xy} + \frac{1}{6}h^2\Delta u_{xy} + \mathcal{O}(h^3) \\
= \quad & \left(1+\frac{1}{6}h^2\right)u_{xy} + \mathcal{O}(h^3) \\
\Rightarrow \quad & \mathcal{O}(h^2)
\end{aligned}
$$

Overall we get a $\mathcal{O}(h^2)$ error term for the cross derivative approximation.

2. This discretisation is isotropic, as the error term incorporates an additional isotropic Laplace operator onto u_{xy}, which we wanted to approximante in the first place.

Problem 4 (Cooking norms)

1. We want to prove the following statement:

$$\frac{1}{\sqrt{n}}\|x\|_2 \overset{1}{\leq} \|x\|_\infty \overset{2}{\leq} \|x\|_2 \overset{3}{\leq} \sqrt{n}\|x\|_\infty$$

We will do this step by step. So at first we prove the first inequality

$$\begin{aligned}
\frac{1}{\sqrt{n}}\|x\|_2 &= \frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n}|x_i|^2} \\
&= \sqrt{\frac{1}{n}\sum_{i=1}^{n}|x_i|^2} \\
&\leq \sqrt{\frac{1}{n}\sum_{i=1}^{n}\max_i \|x_i\|^2} \\
&= \sqrt{\max_i \|x_i\|^2} \\
&= \max_i |x_i|.
\end{aligned}$$

Now we will have a closer look at the second inequality $\|x\|_\infty \leq \|x\|_2$. For this, however we consider now the squared norms, as this does not violate the monotonicity of the norming function:

$$\|x\|_2^2 = \max_{i=1,\dots,n} |x_i|^2 \leq \sum_{i=1}^{n}|x_i|^2 = \|x\|_\infty^2.$$

Now we only need to prove the last inequality, so we compute

$$\begin{aligned}
\|x\|_2 &= \left(\sum_{i=1}^{n}|x_i|^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{n \max_i |x_i|^2} \\
&= \sqrt{n}\sqrt{\max_i |x_i|^2} \\
&= \sqrt{n}\max_i |x_i| \\
&= \sqrt{n}\|x\|_\infty
\end{aligned}$$

which concludes the proof.

2. We want to prove the following statement:

$$\frac{1}{\sqrt{n}}\|x\|_1 \overset{1}{\leq} \|x\|_2 \overset{2}{\leq} \|x\|_1 \overset{3}{\leq} \sqrt{n}\|x\|_2$$

At first, we will prove the third inequality by use of the Cauchy-Schwarz inequality

$$\left(\sum_i x_i y_i\right)^2 \quad \leq \quad \left(\sum_i x_i^2\right) \cdot \left(\sum_i y_i^2\right),$$

so we can compute

$$\|x\|_1 \quad = \quad \sum_{i=1}^{n} 1 \cdot |x_i| \leq \left(\sum_{i=1}^{n} 1^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} = \sqrt{n}\|x\|_2.$$

From this, the first inequality is easily derivable. The biggest problem is now the second inequality. For this we will now consider the vector

$$y \quad = \quad \frac{x}{\|x\|_1} = \frac{x}{\sum_{k=1}^{n} |x_k|}$$

We will now show that $\|y\|_2 \leq 1$ which will help us later.

$$\begin{aligned} \|y\|_2^2 \quad &= \quad \sum_{i=1}^{n} \frac{|x_i|^2}{\left(\sum_{k=1}^{n} |x_k|\right)^2} \\ &= \quad \frac{1}{\sum_{k=1}^{n}} \cdot \sum_{i=1}^{n} \frac{|x_i|^2}{\sum_{k=1}^{n} |x_k|} \\ &\overset{\sum_{k=1}^{n}|x_k| \geq |x_i|}{\leq} \quad \frac{1}{\|x\|_1} \cdot \underbrace{\sum_{i=1}^{n} \frac{|x_i|^2}{|x_i|}}_{\|x\|_1} = 1. \end{aligned}$$

We use this result now in:

$$\|x\|_2 = \|x\|_1 \|y\|_2 \leq \|x\|_1 \cdot 1 = \|x\|_1,$$

what we wanted to show.

Problem 5 (Proving Banach) Let an arbitrary $x_0 \in D$ be given. As $F : D \to D$, the sequence $(x_k)_{k \in \mathbb{N}_0}$ is uniquely determined by $x_{k+1} = F(x_k)$ and for $k \in \mathbb{N}$ it holds:

$$|x_{k+1} - x_k| \quad = \quad |F(x_k) - F(x_{k-1})| \leq L|x_k - x_{k-1}| \leq L^2|x_{k-1} - x_{k-2}| \leq \dots \qquad (*).$$

Also, by iteration of the first approximation it follows for $k \geq n$:

$$|x_{k+1} - x_k| \quad \leq \quad L^{k+1-n}|x_n - x_{n-1}| \qquad (**).$$

and from that also for $m \geq n$:

$$\begin{aligned}
|x_m - x_n| &= |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \ldots (x_{n+1} - x_n)| \\
&\leq \sum_{k=n}^{m-1} |x_{k+1} - x_k| \qquad \text{Triangle-Inequality} \\
&\leq \left(\sum_{k=n}^{m-1} L^{k+1-m} \right) |x_n - x_{n-1}| \\
&\leq \left(\sum_{j=1}^{\infty} L^j \right) |x_n - x_{n-1}| \\
&= L \cdot \left(\sum_{j=0}^{\infty} L^j \right) |x_n - x_{n-1}| \\
&= \frac{L}{1-L} |x_n - x_{n-1}| \\
&\leq \frac{L^n}{1-L} |x_1 - x_0| \qquad \text{see } (*) \text{ or } (**).
\end{aligned}$$

Therefore it holds for $m \geq n$:

$$|x_m - x_n| \quad \leq \quad \frac{L}{1-L}|x_n - x_{n-1}| \leq \frac{L^n}{1-L}|x_1 - x_0| \qquad (***)$$

Due to $L < 1$ it holds that $L^n \to 0$ $(n \to \infty)$ and therefore $(x_n)_{n \in \mathbb{N}_0}$ is a Cauchy sequence that is convergent with the limit x^*. As D is compact, with $***$, $x^* \in D$. F is continuousand therefore the limit x^* is a fixed point of F. x^* is the only fixed point in D. If $x^{**} \neq x^*$ were another fixed point, it would follow:

$$|x^{**} - x^*| = |F(x^{**}) - F(x^*) \leq L|x^{**} - x^*| < |x^{**} - x^*|$$

which is a contradiction. The proposed error approximation follows directly from $(***)$, if $m \to \infty$.
