Numerical Algorithms for Visual Computing 2008/09
Example Solutions for Assignment 3

Problem 1 (The 1-D Spectacle)

1. Let us first have a look at the function $u''(x) = 0$. A proper primitive (Stammfunktion) for this would be $u'(x) = a$ for some constant c and a primitive for this is $u(x) = ax + b$ for another constant b. Now, we need to consider the given boundaries.

   As $u(0) = k_1$, we can calculate $u(0) = a \cdot 0 + b \stackrel{!}{=} k_1$. Therefore, we can set $b = k_1$. Furthermore we have to consider also $u(1) = k_2$. Plugging this into our primitive, we have $u(1) = a \cdot 1 + k_1 = a + k_1 \stackrel{!}{=} k_2$. By setting $a = k_2 - k_1$ we have found a function that satisfies the given constraints, i.e. the exact solution given by the BVP is

   $$u(x) \;=\; (k_2 - k_1)x + k_1.$$

   (a) Is a reasonable problem, as there exist a proper solution for this problem with $u(x) = x + 2$.

   (b) Is a reasonable problem, as there exist a proper solution for this problem with $u(x) = 3x + 1$

   (c) Problem is not reasonable. Considering the primitives for $u''(x) = 0$, i.e. $u'(x) = a$ and $u(x) = ax + b$, the sole boundary condition $u(1) = 4$ results in $a + b = 4$. This however does not give any information on the values a and b, i.e. the conditions on the boundaries have not been given properly.

   (d) Is a reasonable problem, as there exist a proper solution for this problem with $u(x) = x + 3$.

   (e) Problem is not reasonable. The additional constraint $u''(1) = 1$ is in direct violation of the initial condition $u''(x) = 0$.

Problem 2 (Connecting analytical and numerical stuff)

1. First of all, we need to take care of the discretisation of the integral. This can be done, e.g. by use of the rectangle rule (Mittelpunktsregel oder Rechteckregel) of numerical integration, i.e.

   $$\int_a^b f(x)\mathrm{d}x \;\approx\; f\left(\frac{a+b}{2}\right)(b-a)$$

with $(b-a) := \Delta x$ as the grid size of the spatial discretization. Assume we have an appropiate equidistant discretisation of u, i.e. $u_0, \ldots, u_{N+1}$. Let us first make use of the additivity of integrals, i.e.

$$\begin{aligned} E(u) = \int_0^1 [u'(x)]^2 \mathrm{d}x &= \int_{x_0=0}^{x_1} [u'(x)]^2 \mathrm{d}x + \int_{x_1}^{x_2} [u'(x)]^2 \mathrm{d}x + \ldots + \int_{x_N}^{x_{N+1}} [u'(x)]^2 \mathrm{d}x \\ &= \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} [u'(x)]^2 \mathrm{d}x. \end{aligned}$$

By use of the rectangle rule from above we can discretize this even further to

$$\begin{aligned} &= \sum_{i=0}^{N} \Delta x [u'(x_{j+\frac{1}{2}})]^2 \\ &= \sum_{i=0}^{N} \Delta x \left(\frac{u_{i+1} - u_i}{\Delta x} \right)^2 \\ &= \sum_{i=0}^{N} \frac{(u_{i+1} - u_i)^2}{\Delta x}. \end{aligned}$$

2. We are now minimising the discretisation with respect to the unknowns $u_1, \ldots, u_N$. This gives the following system of equations

$$\begin{aligned} \frac{\partial E(u)}{\partial u_1} &= 2\frac{u_1 - u_0}{\Delta x} - 2\frac{u_2 - u_1}{\Delta x} = -2\frac{u_0 - 2u_1 + u_2}{\Delta x} \overset{!}{=} 0 \\ \frac{\partial E(u)}{\partial u_k} &= -2\frac{u_{k-1} - 2u_k + u_{k+1}}{\Delta x} \overset{!}{=} 0 \\ \frac{\partial E(u)}{\partial u_N} &= -2\frac{u_{N-1} - 2u_N + u_{N+1}}{\Delta x} \overset{!}{=} 0. \end{aligned}$$

3. By having a look at the system matrix, we can see a familiar matrix, i.e.

$$\begin{pmatrix} -2 & 1 & & 0 \\ 1 & -2 & 1 & 0 \\ \ddots & \ddots & \ddots & \\ & 1 & -2 & 1 \\ 0 & & 1 & -2 \end{pmatrix} u^\infty = \begin{pmatrix} u_0 \\ 0 \\ \vdots \\ 0 \\ u_{N+1} \end{pmatrix}$$

The system has a unique solution, as the matrix is of full rank, the matrix can be inverted and also the boundaries are properly chosen.

4. This is the same system matrix as we have seen for the direct solution of the Laplace equation $u''(x) = 0$ with fixed boundary points u_0 and u_{N+1}, as the minimization employs a second derivative approximation. This can be directly solved via the Thomas algorithm. So overall, this employs a proper BVP for the Laplace equation.
