Numerical Algorithms for Visual Computing 2008/09
Example Solutions for Assignment 2

Problem 1 (Explicit Coding)

1. For the explicit time marching algorithm, let us consider the already discretized version of the 1-D Laplace equation $\Delta u = 0$, i.e.

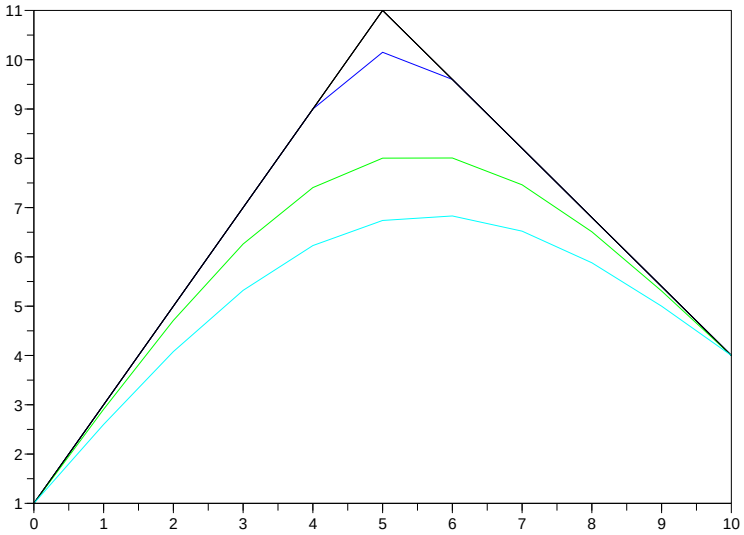
$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\Delta t} &= \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \\ \Leftrightarrow \quad u_j^{n+1} &= \left(1 - 2\frac{\Delta t}{\Delta x^2}\right) u_j^n + \frac{\Delta t}{\Delta x^2} u_{j-1}^n + \frac{\Delta t}{\Delta x^2} u_{j+1}^n \end{aligned}$$

   By considering our initial vector

$$u^0 = (1, 3, 5, 7, 9, 11, 9.6, 8.2, 6.8, 5.4, 4)^\top,$$

   we have the following results after 1, 10 and 20 iterations respectively:

$$\begin{aligned} u^1 &= (1, 3, 5, 7, 9, 10.15, 9.6, 8.2, 6.8, 5.4, 4)^\top, \\ u^{10} &= (1, 2.9, 4.7, 6.3, 7.4, 8, 8, 7.5, 6.5, 5.3, 4)^\top \\ u^{20} &= (1, 2.6, 4.1, 5.3, 6.2, 6.7, 6.8, 6.5, 5.9, 5, 4)^\top \end{aligned}$$



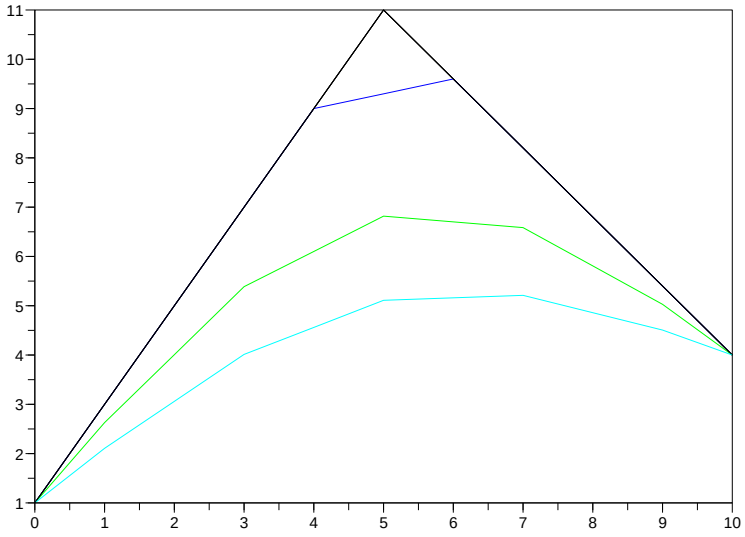
2. The Jacobi iteration scheme can be derived quite easily from

$$\begin{aligned} 0 &= \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \\ \Leftrightarrow \quad u_j^n &= \frac{u_{j-1}^n + u_{j+1}^n}{\Delta x^2} \end{aligned}$$

In general, this boils down to the averaging of both neighbouring pixels. From this we can get the results

$$\begin{aligned} u^1 &= (1, 3, 5, 7, 9, 9.3, 9.6, 8.2, 6.8, 5.4, 4)^\top, \\ u^{10} &= (1, 2.6, 4, 5.4, 6.1, 6.8, 6.7, 6.6, 5.8, 5, 4)^\top \\ u^{20} &= (1, 2.1, 3.1, 4, 4.6, 5.1, 5.2, 5.2, 4.9, 4.5)^\top. \end{aligned}$$

For the entire code implementation, see also the scilab file.



Problem 2 (Direct Coding) The crucial part of this exercise is the derivation of the sought linear system of equations. For our given vector, we know that the boundary values u_0 and u_{10} should remain constant, and from our observation of the first exercise, we would expect a linear function as a result. From this we can conclude, that out of 11 possible equations, 2 are already

fixed. We only have to concentrate on the unknowns $u_1, \ldots, u_9$. By use of the equation

$$0 \;=\; u_{j+1}^k - 2u_j^k + u_{j-1}^k \qquad j = 1, \ldots, 9,$$

we have the sought system of equations already. We only need to take care at the boundaries, i.e.

$$\begin{aligned} -2u_1^n + u_2^n &= -u_0^n \\ -2u_9^n + u_8^n &= -u_{10}^n, \end{aligned}$$

which gives us the system

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} u^\infty \;=\; \begin{pmatrix} -u_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -u_{10} \end{pmatrix}$$

This can be easily solved via the Thomas algorithm with the result

$$u^\infty \;=\; (1, 1.3, 1.6, 1.9, 2.2, 2.5, 2.8, 3.1, 3.4, 3.7, 4)^\top,$$

which is the linear function we have been looking for. (see the scilab-file for more details on the programming side).

Problem 3 (Implicit Coding) For this problem, we consider the implicit Laplace discretisation

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} \;=\; \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}.$$

This gives us after some computation

$$u_j^n \;=\; \left(1 + 2\frac{\Delta t}{\Delta x^2}\right) u_j^{n+1} - \frac{\Delta t}{\Delta x^2} u_{j-1}^{n+1} - \frac{\Delta t}{\Delta x^2} u_{j+1}^{n+1}.$$

Again, we have to take extra precaution at the boundaries, so our implicit scheme for our problem is as follows:

$$\left(I - \frac{\Delta t}{\Delta x^2} \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \right) u^{n+1} = \begin{pmatrix} u_1 + \frac{\Delta t}{\Delta x^2} u_0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 + \frac{\Delta t}{\Delta x^2} u_{10} \end{pmatrix}$$

As for the experiments, one can see that one can use time step sizes of arbitrary length for the Δt in the implicit scheme, whereas for the explicit scheme, $\Delta t \leq 0.5$ in order to be stable, or else the scheme diverges.

Problem 4 (Jacobi Strikes Back)

1. Similar to the central difference approximation of the second derivative u_{xx} at point $j\Delta x$, we do the same thing for the backward difference approximation

$$u_{xx}(j\Delta x) \approx \frac{u_x(j\Delta x) - u_x((j-1)\Delta x)}{\Delta x}$$

The values $u_x(j\Delta x)$ and $u_x((j-1)\Delta x)$ can be approximated by

$$\frac{u_j - u_{j-1}}{\Delta x} \quad \text{and} \quad \frac{u_{j-1} - u_{j-2}}{\Delta x}$$

Then

$$\begin{aligned} u_{xx}(j\Delta x) &\approx \frac{u_x(j\Delta x) - u_x((j-1)\Delta x)}{\Delta x} \\ &\approx \frac{\frac{u_j - u_{j-1}}{\Delta x} - \frac{u_{j-1} - u_{j-2}}{\Delta x}}{\Delta x^2} \\ &\approx \frac{u_j - 2u_{j-1} + u_{j-2}}{\Delta x^2} \end{aligned}$$

2. For the one-dimensional Laplace-equation

$$u_{xx} = 0$$

one can take the finite difference approximation obtained from exercise 1a and solve for u_j

$$\frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{\Delta x^2} = 0$$

From that we can obtain the iterative scheme

$$u_j^{(n+1)} = 2u_{j-1}^{(n)} - u_{j-2}^{(n)}.$$

3. This iterative scheme is not really useful, as the scheme is biased in that sense that it depends on 2 points before the evaluation point u_j. Therefore is only a shift into the right direction possible. Also one needs different boundary conditions on the left end of the interval, i.e. some kind of second boundary condition. However, one does not need any boundary condition on the right interval end.
