Numerical Algorithms for Visual Computing 2008/09
Example Solutions for Assignment 1

Problem 1 (Types of PDEs) First of all, let us have a closer look at what we are doing here. In general, the differential equation that we see, can be described in terms of a conic section. For more information, please refer to the mathematics for computer science lectures on quadrics. One see a differential equation in terms of a quadric by computing the following

$$(x, y, 1)\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}(x, y, 1)^\top \quad = \quad g$$

which happens to be, after some calculation, the equation

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = g,$$

and for sake of simplicity, the factors in front of b,d,e are omitted (they can be incorporated into them). However, we are mostly interested in the upper 2×2-matrix and its resulting quadric

$$(x, 1)\begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}(x, 1)^\top = ax^2 + bx + c.$$

If we now have a look at the discriminant

$$\begin{vmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{vmatrix} = ac - \frac{1}{4}b^2 \stackrel{!}{=} 0,$$

we can see that this our sought relation. Furthermore, if we have a look back at the equation $ax^2 + bx + c$, this equation is quite well known from school, it has two solutions

$$x_{1,2} \quad = \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

and, depending on the sign of the square root, has 0, 1 or 2 solutions. The equation $b^2 - 4ac > 0$ gives two solutions, equal zero has only one solution if it is negative, it has no real solution. Let us come back to our problem, this gives an analogon for the categorisation of the differential equation. In case, the discriminant is equal to zero, we have a parabolic equation, in case we have no solution, we have an elliptic equation and for two solutions, we have a hyperbolic equation.

1. $u_t = u_{xx}$ (Diffusion equation)
   We have $a = 1$, $b = c = d = f = g = 0$ and $e = -1$, therefore, $b^2 - 4ac = 0$.
   Parabolic, Second order, linear

2. $u_{tt} = u_{xx}$ (Wave equation)
   $a = 1$, $c = -1$, $b = d = e = f = g = 0$, i.e. $b^2 - 4ac = 4 > 0$.
   Hyperbolic, Second order, linear

3. $u_{xx} + u_{yy} = 0$ (Laplace equation)
   $a = c = 1$, $b = d = e = f = g = 0$, i.e. $b^2 - 4ac = -4 < 0$.
   Elliptic, Second order, linear

4. $xu_x + yu_y + u^2 = 0$
   $a = b = c = 0$,$d = x$,$e = y$,$f = u$, i.e. $b^2 - 4ac = 0$.
   Parabolic, First order, non-linear. The non-linearity comes from the coefficient f, which squares the u-term. The terms x and y are so-called variable coefficients and in general this does not imply non-linearity (This was mixed up in the exercise group, I'm afraid).

5. Special entry $yu_{xx} + u_{yy} = 0$. $a = 1$,$c = y$, i.e. $b^2 - 4ac = -4y$. This one depends on the variable y, i.e. for $y > 0$ elliptic, $y = 0$ parabolic, $y < 0$ hyperbolic
   Second order, nonlinear.

Problem 2 (Taylor expansions and difference schemes) There are two possibilities in order to compute the sought difference scheme.

Method 1:
Analogously to the scheme (2.6) in the script, we compute $u''(j\Delta x)$ by using an approximation with use of the central difference scheme. For this, we use the approximations

$$\begin{aligned} u''((j+1)\Delta x) &\approx \frac{u_{j+2} - u_j}{2\Delta x} \\ u''((j-1)\Delta x) &\approx \frac{u_j - u_{j-2}}{2\Delta x} \end{aligned}$$

Then, we can combine this

$$\begin{aligned} u''(j\Delta x) &\approx \frac{u''((j+1)\Delta x) - u''((j-1)\Delta x)}{2\Delta x} \\ &\approx \frac{\frac{u_{j+2}-u_j}{2\Delta x} - \frac{u_j - u_{j-2}}{2\Delta x}}{2\Delta x} \\ &= \frac{u_{j-2} - 2u_j + u_{j+2}}{4\Delta x} \end{aligned}$$

<u>Method 2: Taylor-Expansioning</u>

$$\begin{aligned} u(j+2\Delta x) = u_{j+2} &= u_j + 2\Delta x u_j' + 2\Delta x^2 u_j'' + \frac{4}{3}\Delta x^3 u_j''' + \frac{2}{3}\Delta x^4 u_j'''' + \mathcal{O}(\Delta x^5) \\ u_j &= u_j \\ u(j-2\Delta x) = u_{j-2} &= u_j - 2\Delta x u_j' + 2\Delta x^2 u_j'' - \frac{4}{3}\Delta x^3 u_j''' + \frac{2}{3}\Delta x^4 u_j'''' + \mathcal{O}(\Delta x^5) \end{aligned}$$

Now, in order to compute the sought derivative scheme, we need two additional information for our scheme

$$\begin{aligned} u_j'' &= 0 \cdot u_j + 0 \cdot u_j' + 1 \cdot u_j'' \\ u_j'' &= \alpha u_{j-2} + \beta u_j + \gamma u_{j+2} \end{aligned}$$

By inserting into the Taylor expansions into the equations, we can get

$$\begin{aligned} & u_j \underbrace{(\alpha + \beta + \gamma)}_{=0} \\ + \; & u_j' \Delta x \underbrace{(-2\alpha + 2\gamma)}_{=0} \\ + \; & u_j'' \Delta x^2 \underbrace{(2\alpha + 2\gamma)}_{=0} \\ + \; & u_j''' \Delta x^3 (\frac{4}{3}\gamma - \frac{4}{3}\alpha) \\ + \; & u_j'''' \Delta x^4 (\frac{2}{3}\alpha + \frac{2}{3}\gamma) + \mathcal{O}(h^5) \end{aligned}$$

Note that we need the Δx^3 and Δx^4 factors later when it comes to computing the local truncation error. This gives $\alpha = \gamma = \frac{1}{4\Delta x^2}$ and $\beta = -\frac{2}{4\Delta x^2}$, as we have computed above. Now for the local truncation error, we compute the first non-vanishing term. The Δx^3-term however vanishes, whereas the fourth order derivative does not vanish, i.e.

$$L_{\Delta x}(u) \;=\; u'' + \frac{1}{3}u_j''''\Delta x^2 + \mathcal{O}(\Delta x^3) = u'' + \mathcal{O}(\Delta x^2) = \mathcal{O}(\Delta x^2)$$

Problem 3 (Big $\mathcal{O}$ I) Remember:

$$\varphi(h) = \mathcal{O}(h^p) \quad \Leftrightarrow \quad \lim_{h\to 0} \frac{\varphi(h)}{h^p} = C < \infty.$$

1. $\mathcal{O}(h^p) + \mathcal{O}(h^q) = \mathcal{O}(h^p)$
   Let $\varphi_1(h) = \mathcal{O}(h^p)$ and $\varphi_2(h) = \mathcal{O}(h^q)$. This can be introduced into the definition

$$\begin{aligned}
\lim_{h\to 0} \frac{\varphi_1(h) + \varphi_2(h)}{h^p} &= \lim_{h\to 0} \frac{\varphi_1(h)}{h^p} + \frac{\varphi_2(h)}{h^p} \\
&= \lim_{h\to 0} \frac{\varphi_1(h)}{h^p} + \lim_{h\to 0} \frac{\varphi_2(h)}{h^p} \\
&= \lim_{h\to 0} \frac{\varphi_1(h)}{h^p} + \lim_{h\to 0} \frac{\varphi_2(h)}{h^q \cdot h^{p-q}} \\
&= \underbrace{\lim_{h\to 0} \frac{\varphi_1(h)}{h^p}}_{\mathcal{O}(h^p)} + \underbrace{\lim_{h\to 0} \frac{\varphi_2(h)}{h^q}}_{\mathcal{O}(h^q)} \cdot \underbrace{\lim_{h\to 0} \frac{1}{h^{p-q}}}_{\to 0} \\
&= |c_1| + \lim_{h\to 0} \frac{1}{h^{p-q}} |c_2| = |c_1| \\
&= \mathcal{O}(h^p)
\end{aligned}$$

2. $\mathcal{O}(h^p) \cdot \mathcal{O}(h^q) = \mathcal{O}(h^{p+q})$

$$\begin{aligned}
\lim_{h\to 0} \frac{\varphi_1(h) \cdot \varphi_2(h)}{h^{p+q}} &= \lim_{h\to 0} \frac{\varphi_1(h)}{h^p} \cdot \frac{\varphi_2(h)}{h^q} \\
&= \lim_{h\to 0} \frac{\varphi_1(h)}{h^p} \cdot \lim_{h\to 0} \frac{\varphi_2(h)}{h^q} \\
&= |c_1| \cdot |c_2| = |c_1 \cdot c_2| < \infty
\end{aligned}$$

3. $\mathcal{O}(h^p) - \mathcal{O}(h^p) = \mathcal{O}(h^p)$

$$\begin{aligned}
\lim_{h\to 0} \frac{f(h) - g(h)}{h^p} &= \lim_{h\to 0} \frac{f(h)}{h^p} - \lim_{h\to 0} \frac{g(h)}{h^p} \\
&= \frac{f(h) - g(h)}{h^p} = |c_1| - |c_2| < \infty
\end{aligned}$$

4. As seen in the script, $\frac{1}{\mathcal{O}(h^p)}$ does not exist.

Problem 4 (Big $\mathcal{O}$ II)

$$\begin{aligned} a(h) &= h + h^2 + 10^{20}h^3 \\ b_1(h) &= h + h^2 + 10^{20}h^3 + 10^{-100}h^4 \\ b_2(h) &= -h - h^2 + 10^{20}h^3 + 10^{-100}h^4. \end{aligned}$$

1. $a(h) \cdot b(h)$

$$\begin{aligned} a(h) \cdot b(h) &= (h + h^2 + 10^{20}h^3) \cdot (h + h^2 + 10^{20}h^3 + 10^{-100}h^4) \\ &= h^2 + 2h^3 + 10^{20}h^4 + 10^{-100}h^5 + h^3 + h^4 \\ &+ 10^{20}h^5 + 10^{-100}h^6 + 10^{20}h^4 + 10^{20}h^5 + 10^{40}h^6 + 10^{-80}h^7 \\ &= h^2 + 2h^3 + (2 \cdot 10^{20} + 1)h^4 + (2 \cdot 10^{20} + 10^{-100})h^5 \\ &+ (10^{-100} + 10^{40})h^6 + 10^{-80}h^7 \\ &\overset{\text{see 3a}}{=} h^2 + \mathcal{O}(h^2) = \mathcal{O}(h^2) \end{aligned}$$

2. $b_1(h) - a(h)$

$$\begin{aligned} b_1(h) - a(h) &= h + h^2 + 10^{20}h^3 + 10^{-100}h^4 - h + h^2 + 10^{20}h^3 \\ &= 10^{-100}h^4 = \mathcal{O}(h^4) \end{aligned}$$

3. $b_1(h) + b_2(h)$

$$\begin{aligned} b_1(h) + b_2(h) &= h + h^2 + 10^{20}h^3 + 10^{-100}h^4 - h - h^2 + 10^{20}h^3 + 10^{20}h^3 \\ &= 2 \cdot 10^{20}h^3 + 2 \cdot 10^{-100}h^4 \overset{\text{see 3a}}{=} \mathcal{O}(h^3) \end{aligned}$$