NUMERICAL ALGORITHMS FOR VISUAL COMPUTING II

ASSIGNMENT 8

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8.1   Shake That!

a. As you like, the inline source code:

```
function upwind()
  // constructing signal as follows:
  // sig   00000 11111 11111 00000 00000 ...
  // idx  -4   0 1     6     11    16     ...
  signal = zeros(50,1);
  signal(5:15,1) = 1;

  s  = 20;      // steps
  a  = 2.0;     // parameter a
  dx = 0.1;     // delta x
  dt = 0.025;   // delta t

  // plot original signal
  plot2d(1:50,signal,style=1)

  for i=1:s
    // build boundary-zero-ed signal and shift one to left
    n = size(signal,1)
    tmp = zeros(n+2,1)
    tmp(2:n+1,1) = signal(1:n,1);

    for j=1:n
     signal(j,1) = dt * a / dx * (tmp(j,1) - tmp(j+1,1)) + tmp(j+1,1)
    end

    // plot every 5 iterations
    if modulo(i,5) == 0
      plot2d(1:50,signal,style=i/5)
    end
  end
endfunction
```
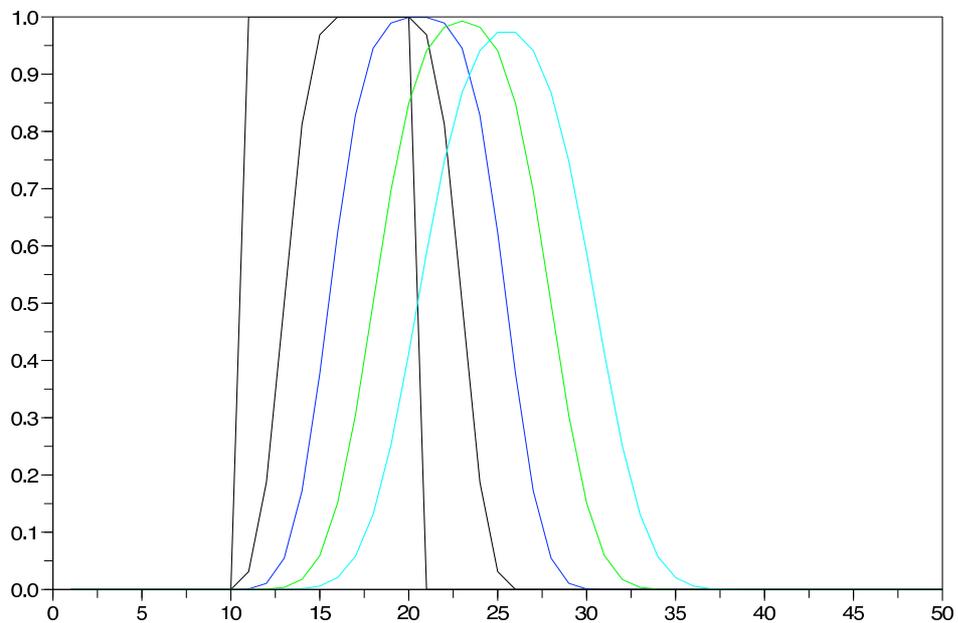
The plot shows every fifth iteration and also includes the initial signal.

b. We know from the lecture

$$u_i \approx u(i\Delta x) \stackrel{(2.12)}{=} u(x)$$

$$u_{i-1} \approx u((i-1)\Delta x) \stackrel{(2.12)}{=} u(x - \Delta x)$$

$$u^n \approx u(n\Delta t) \stackrel{(2.12)}{=} u(t)$$

$$u^{n+1} \approx u((n+1)\Delta t) \stackrel{(2.12)}{\approx} u(t + \Delta t)$$

Let's compute the Taylor expansions:

$$u(x) = u(x)$$

$$u(x - \Delta x) = u(x) + [(x - \Delta x) - x]u'(x) + \frac{[(x + \Delta x) - x]^2}{2}u''(x)$$

$$+ \frac{[(x - \Delta x) - x]^3}{6}u'''(x) + \frac{[(x + \Delta x) - x]^4}{24}u''''(x)$$

$$+ \frac{[(x - \Delta x) - x]^5}{120}u'''''(x) + \mathcal{O}(\Delta x^6)$$

$$u(t) = u(t)$$

$$u(t + \Delta t) = u(t) + [(t + \Delta t) - t]u'(t) + \frac{[(t + \Delta t) - t]^2}{2}u''(t)$$

$$+ \frac{[(t + \Delta t) - t]^3}{6}u'''(t) + \frac{[(t + \Delta t) - t]^4}{24}u''''(t)$$

$$+ \frac{[(t + \Delta t) - t]^5}{120}u'''''(t) + \mathcal{O}(\Delta x^6)$$

Let's compute the local truncation error (neglecting the arguments x and t, respectively):

$$L(u) = \frac{u_i^{n+1} - u_i^n}{\Delta t} + a\,\frac{u_i^n - u_{i-1}^n}{\Delta x} - 0$$

$$
\begin{aligned}
L(u) \;=\;& \frac{1}{\Delta t}\left[u + \Delta t\, u_t + \frac{\Delta t^2}{2}u_{tt} + \frac{\Delta t^3}{6}u_{ttt}\right.\\
&\left.+\frac{\Delta t^4}{24}u_{tttt} + \frac{\Delta t^5}{120}u_{ttttt} + \mathcal{O}(\Delta t^6)\right]\\
&-\frac{u}{\Delta t} + a\left(\frac{u}{\Delta x}\right.\\
&-\frac{1}{\Delta x}\left[u - \Delta x\, u_x + \frac{\Delta x^2}{2}u_{xx} - \frac{\Delta x^3}{6}u_{xxx}\right.\\
&\left.\left.+\frac{\Delta x^4}{24}u_{xxxx} - \frac{\Delta x^5}{120}u_{xxxxx} + \mathcal{O}(\Delta x^6)\right]\right)\\
\;=\;& \underbrace{(u_t + au_x)}_{=0} + \underbrace{\frac{\Delta t}{2}u_{tt} + \frac{\Delta t^2}{6}u_{ttt} + \frac{\Delta t^3}{24}u_{tttt} + \frac{\Delta t^4}{120}u_{ttttt} + \mathcal{O}(\Delta t^5)}_{\mathcal{O}(\Delta t)}\\
& + \underbrace{a\left(-\frac{\Delta x}{2}u_{xx} + \frac{\Delta x^2}{6}u_{xxx} - \frac{\Delta x^3}{24}u_{xxxx} + \frac{\Delta x^4}{120}u_{xxxxx} + \mathcal{O}(\Delta x^5)\right)}_{\mathcal{O}(\Delta x)}
\end{aligned}
$$

$$
\left[\;=\; \left(\frac{\Delta t}{2}u_{tt} - a\frac{\Delta x}{2}u_{xx}\right) + \mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta x)^2)\right]
$$

$$
=\; \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x)
$$

c. We have
$$
u_t + au_x = 0 \Leftrightarrow u_t = -au_x.
$$

$$
\begin{aligned}
u_{tt} &= (-au_x)_t\\
&= -a(u_x)_t\\
u_{xx} &= \left(-\frac{1}{a}u_t\right)_x\\
&= -\frac{1}{a}(u_t)_x\\
&= -\frac{1}{a}(u_x)_t
\end{aligned}
$$

Now, let's check the relationship:

$$
\begin{aligned}
u_{tt} &= a^2 u_{xx}\\
-a(u_x)_t &= a^2\left(-\frac{1}{a}\right)(u_x)_t\\
a &= a \quad \checkmark
\end{aligned}
$$

d. We use the second last line of the computation of $L(u)$ which reads

$$
\begin{aligned}
L(u) \;=\;& \frac{\Delta t}{2}u_{tt} - a\frac{\Delta x}{2}u_{xx} + \mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta x)^2)\\[4pt]
\stackrel{\text{Hint 2}}{=}\;& \frac{\Delta t}{2}u_{tt} - a\frac{\Delta x}{2}u_{xx} + \mathcal{O}((\Delta x)^2)\\[4pt]
\stackrel{\text{Hint 3}}{=}\;& \frac{\Delta t}{2}(a^2 u_{xx}) - a\frac{\Delta x}{2}u_{xx} + \mathcal{O}((\Delta x)^2)\\[4pt]
=\;& u_{xx}\left(\frac{a^2\,\Delta t - a\,\Delta x}{2}\right) + \mathcal{O}((\Delta x)^2)\\[4pt]
\stackrel{\text{Hint 2}}{=}\;& u_{xx}\left(\frac{a^2\,(\lambda\Delta x) - a\Delta x}{2}\right) + \mathcal{O}((\Delta x)^2)\\[4pt]
=\;& u_{xx}\underbrace{\left(\frac{a\,\Delta x}{2}\,(a\,\lambda - 1)\right)}_{\alpha(\Delta x)} + \mathcal{O}((\Delta x)^2)
\end{aligned}
$$

$\square$

The term $u_{xx}\alpha(\Delta x)$ leads to a smoothing of the signal which depends heavily on a.

e. We derived in the previous two parts of this exercise the first level order accurate solution with the leading error term. If we want to modify the given equation of the upwind scheme in such a way that it is second-order accurate, we have to build an equation so that the first order error term cancels out.

In this part of the exercise, we have to prove that the following equation is second-order accurate:

$$
\begin{aligned}
u_t + au_x \;&=\; \frac{a\Delta x}{2}(1 - a\lambda)u_{xx}\\[4pt]
u_t + au_x + \frac{a\Delta x}{2}(a\lambda - 1)u_{xx} \;&=\; 0
\end{aligned}
$$

With the result from the previous exercise we get immediately the second-order accuracy.

$$
L(u) \;=\; \underbrace{\left(u_t + au_x + u_{xx}\left(\frac{a\,\Delta x}{2}\,(a\,\lambda - 1)\right)\right)}_{=0} + \mathcal{O}((\Delta x)^2)
$$

$\square$

8.2   No Time To Chill

a. We know from the lecture that

$$u_i^{n+1} = u_i^n - a\frac{\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$$

can be expressed by a scheme function $\mathcal{H}$

$$\mathcal{H}(u_{i-1}^n, u_i^n) = u_i^n - a\frac{\Delta t}{\Delta x}(u_i^n - u_{i-1}^n).$$

A scheme is monotone if

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial u_{i-1}^n} &= a\frac{\Delta t}{\Delta x} \geq 0 \\
\frac{\partial \mathcal{H}}{\partial u_i^n} &= 1 - a\frac{\Delta t}{\Delta x} \geq 0 \\
&\Leftrightarrow \Delta t \leq \frac{\Delta x}{a}
\end{aligned}$$

(a)

$$\begin{aligned}
\Delta t &\leq \frac{\Delta x}{a_1} = \frac{2}{5} \geq 0 \\
a_1\frac{\Delta t}{\Delta x} &\leq 0.5 \cdot \frac{0.4}{0.2} = 1 \geq 0 \quad \checkmark
\end{aligned}$$

For $\Delta t \leq \frac{2}{5} = 0.4$, the scheme is monotone.

(b)

$$\begin{aligned}
\Delta t &\leq \frac{\Delta x}{a_2} = \frac{1}{25} \geq 0 \\
a_2\frac{\Delta t}{\Delta x} &\leq 5 \cdot \frac{0.04}{0.2} = 1 \geq 0 \quad \checkmark
\end{aligned}$$

For $\Delta t \leq \frac{1}{25} = 0.04$, the scheme is monotone.

b. For the resulting conditions, we get a discrete minimum-maximum-principle for $\Delta t \leq \frac{2}{5}$ and $\Delta t \leq \frac{1}{25}$, respectively, since the monotony property implies for hyperbolic PDEs an upper and lower bound in the L_∞-norm. This directly implies stability.