Numerical Algorithms for Visual Computing II

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Assignment 8 (2 Exercises) - The Scooter Package - Hyper, Hyper

Consider on this sheet the *hyperbolic* linear advection equation

$$u_t + au_x \quad = \quad 0, \qquad a > 0, \tag{1}$$

with $u(x, 0) := u_0(x)$. The upwind scheme for solving (1) reads as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} \quad = \quad 0 \tag{2}$$

Exercise No. 1 – Shake That! (6+4+2+6+2 pts)

a) Write a program solving (1) via (2). Use the parameters $a = 2, \Delta x = 0.1$, $\Delta t = 0.025$, and perform 20 time steps iterating

$$U_i^0 \quad = \quad \begin{cases} 1, & i = 0, \dots, 9, \\ 0, & \text{else}, \end{cases} \tag{3}$$

where $U_i^0 \approx u(i\Delta x, 0)$. Plot your result.

b) Compute the local truncation error of the upwind scheme, compare §2.

Hint: You need Taylor series expansions in the time variable t as well as in the space variable x. These work both as usual.

c) Verify the relationship

$$u_{tt} \quad = \quad a^2 u_{xx} \tag{4}$$

by use of the PDE (1).

d) Compute the leading error term in the format

$$L(u) \quad = \quad \alpha(\Delta x) \cdot u_{xx} + \mathcal{O}((\Delta x)^2), \tag{5}$$

i.e. derive the function $\alpha(\Delta x)$.

Hints:

- $\lambda := \frac{\Delta t}{\Delta x} = \text{constant}$
- $\mathcal{O}((\Delta t)^2) = \mathcal{O}((\Delta x)^2)$, since $\Delta t = \lambda \cdot \Delta x$
- Use equation (4)

Can you explain the shape of your numerical solution obtained in part (a) by the form of $L(u)$?

e) The so-called modified equation of the upwind scheme (2) for approximating (1) reads as the PDE

$$u_t + au_x \quad = \quad \frac{a\Delta x}{2}(1 - a\lambda)u_{xx}. \tag{6}$$

Verify, that the scheme (2) gives a second-order accurate solution of (6).

Hint: You need to make use of (6) instead of (1) within the computation.

Exercise No. 2 – No Time To Chill (4+6 pts)

(a) Let

a) $a_1 = \frac{1}{2}$,

b) $a_2 = 5$,

and $\Delta x = 0.2$. Derive corresponding CFL-conditions ensuring the monotony of (2).

(b) Discuss, if the resulting condition is a stability condition or not.