Numerical Algorithms for Visual Computing II

PD Dr. Michael Breuß and M.Sc. Kai Uwe Hagenburg

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Assignment 6 (7 Exercises) - The CG global illumination

Exercise No. 1 – Proving Stuff I (3 points)

Show, that $F(x)$ from equation (10.5), i.e.

$$F(x) = \frac{1}{2}(Ax, x)_2 - (b, x)_2$$

is strictly convex and has a unique minimum. You may assume, that A is a SPD matrix.

Exercise No. 2 – Proving Stuff II (3 points)

Prove Corollary 10.5:

The global minimum of $f_{x,p}$, i.e. the minimum of F, starting in x and searching along $x + \lambda p$, is given by

$$\lambda_{opt} = \frac{(r, p)_2}{(Ap, p)_2},$$

where $r := b - Ax$.

Exercise No. 3 – Scheming Schemes (3 points)

Try to formulate the steepest descent scheme in terms of an iterative scheme

$$\phi(x, b) = Mx + Nb.$$

Is the scheme consistent, or linear?

Exercise No. 4 – A Descent into the Maelstrom (6 points)

Solve the system $Ax = b$ by the steepest descent scheme, where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix},$$
$$b = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$x_0 = \begin{pmatrix} 4 \\ \sqrt{1.8} \end{pmatrix}$$

1. Give a table of your iterates after $5, 10, 15, \ldots, 70$ iteration steps, together with the corresponding error in the 2-norm.
2. Visualise the iterates of the first 10 steps by plotting them in the x-y-plane together with the level sets of $F(x, y)$ on which they are located. Describe the behaviour of the solver.

Exercise No. 5 – Descending even further (3 points)

1. Show that the iterates x_m of the gradient descent scheme are optimal w.r.t. the direction $r_{m-1} = b - Ax_{m-1}$.
2. Classify the gradient descent scheme in terms of a projection method.

Exercise No. 6 – A Descent into the Maelstrom with CG (2 points)

Repeat the process of Exercise 4, this time using the CG-scheme. Play around with the initial point x_0. What do you observe?

Exercise No. 7 – Comparison of different methods (10 points)

Consider the Poisson equation

$$\begin{aligned} -u'' &= f(x) \qquad x \in \Omega \subset \mathbb{R} \\ u(x) &= 0 \qquad x \in \partial\Omega = \{0, 1\} \end{aligned}$$

For $\Omega = (0, 1)$ being the unit sphere, we discretise this line with 7 discretisation points. This gives us the step size $h = \frac{1}{8}$. You may use the discretisation of the Laplacian, as it was given in (2.6), i.e.

$$-u'' = \frac{2u_i - u_{i-1} - u_{i+1}}{h^2}$$

For the right hand side of the arising system $Au_h = f_h$, use the function $f(x) = 2x$. Compare in the following the results with the following methods:

- Jacobi
- SOR
- CG

Measure (i) the number of iterates and (ii) the computation time needed to achieve machine accuracy. Also investigate the 2-norm of the residuals arising during the process. What do you observe?