Numerical Algorithms for Visual Computing II

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Assignment 4 (5 Exercises) – Mind cooking on diverse stuff

Exercise No. 1 – Shift invariance of the Laplace operator

Prove that the Laplace equation is shift invariant under translation, i.e.

$$\begin{aligned} x' &= x + a \\ y' &= y + b \end{aligned}$$

such that

$$u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'}$$

holds.
Hint: Consider x, y as a mapping depending on x' and y' respectively. **(6 pts)**

Exercise No. 2 – What's the Matrix, what's the matrix?

Consider the method (5.3) for the approximation of the Poisson problem. Let two *linear orderings* (in analogy to the lexicographical ordering) of the computational nodes of a grid be given as indicated by the following tables:

(a)

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

(b)

1	3	6	10
2	5	9	13
4	8	12	15
7	11	14	16

Sketch the structure of the arising matrices setting up the scheme (5.3) by use of a „$*$“ for potential non-zero entries. **(6 pts)**

Exercise No. 3 – Crossing Derivatives

Consider the following discretisation of the cross derivative u_{xy}:

$$(u_{xy_C})_{i,j} \quad = \quad \frac{1}{4h^2}(u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1})$$

1. Compute the order of consistency of this discretisation.
2. Is this discretisation isotropic or anisotropic?

<u>General Hint:</u> The 2-D Taylor series about a point $(x, y) = (x_0, y_0)$ can be computed by successively performing two 1-D Taylor series. This yields

$$u(x,y) \quad = \quad \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!}\frac{1}{l!}\left(\frac{\partial}{\partial x^k}\frac{\partial}{\partial y^l}f(x_0,y_0)\right)(x-x_0)^k(y-y_0)^l.$$

In order to simplify the computation, one often reformulates this 2-D Taylor series in terms of sums that contain partial derivatives of the same order s:

$$u(x,y) \quad = \quad \sum_{s=0}^{\infty}\frac{1}{s!}\sum_{k=0}^{s}\binom{s}{k}\left(\frac{\partial}{\partial x^k}\frac{\partial}{\partial y^{s-k}}f(x_0,y_0)\right)(x-x_0)^k(y-y_0)^{s-k}.$$

(6 pts)

Exercise No. 4 – Cooking norms

Prove the following inequalitites for vector norms

1. $\frac{1}{\sqrt{n}}\|x\|_2 \le \|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$
2. $\frac{1}{\sqrt{n}}\|x\|_1 \le \|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$

(6 pts)

Exercise No. 5 – Proving Banach

Prove the a priori statement

$$\|x_n - x\| \quad \le \quad \frac{q^n}{1-q}\|x_1 - x_0\|$$

of Theorem 7.11. **(6 pts)**