On Robust Estimation and Smoothing with Spatial and Tonal Kernels

Jan Hendrik Dithmar
jadi5003@stud.uni-saarland.de

Based on
"On Robust Estimation and Smoothing with Spatial and Tonal Kernels"
by P. Mrázek, J. Weickert and A. Bruhn.

Motivation

- Image denoising or simplification are well established nowadays. There exist several methods.
- Examples of different methods:
  - Statistical estimation
  - Histogram operations
  - Local M-smoothers
  - Bilateral filtering
  - ...
- Advantages of the various approaches and the relations between the different methods are only partially understood.
- What can be done to understand the different approaches in a better way?

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

Statistical estimation

- Assume, u being an *unknown (constant)* signal.
- We have noisy samples f_j

$$f_j = u + n_j \qquad (j = 1, ..., N)$$

where n stands for the noise.

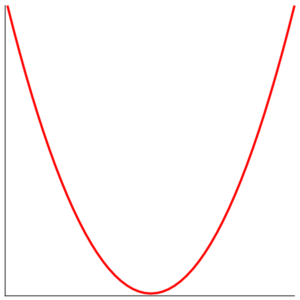
- Goal: Estimate u ($\rightarrow$ M-Estimate).
- M-Estimate can be found by minimising the energy

$$E(u) = \sum_{j=1}^{N} \Psi\left(|u - f_j|^2\right)$$

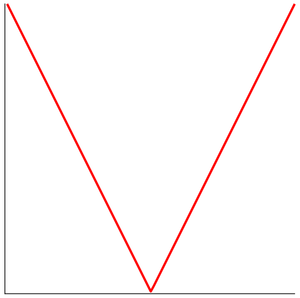
where Ψ is some increasing function (also called error norm or penaliser).

Examples of error norms for M-estimators

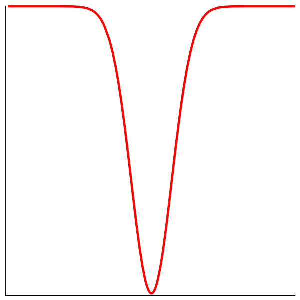
- $\Psi(s^2) = s^2$
  - minimises the L^2-distance $E(u) = \sum_{j=1}^{N} |u - f_j|^2$
  - gives the arithmetic mean (good for removing Gaussian noise)



- $\Psi(s^2) = |s|$
  - minimises the L^1-distance $E(u) = \sum_{j=1}^{N} |u - f_j|$
  - gives the median (good for removing impulse noise)

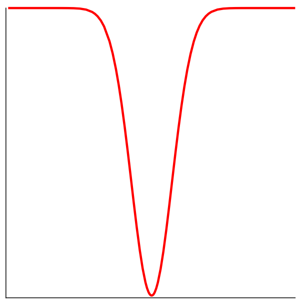


- $\Psi(s^2) = 1 - e^{-s^2/\lambda^2}$
  - mode approximation (influence of outliers is very much reduced)



What is the problem of this approach?

- $\Psi(s^2) = 1 - e^{-s^2/\lambda^2}$
  - mode approximation (influence of outliers is very much reduced)



What is the problem of this approach?

We only get one estimate!

Histogram operations

- In image analysis, the data f_j is measured at positions x_j.
- Usually, we want a solution $\mathbf{u} = (u_i)_{i=1,\ldots,N}$ where each output value u_i belongs to the position x_i.
- Construction of M-estimates u_i by minimising

$$E(\mathbf{u}) = \sum_{i=1}^{N} \sum_{j=1}^{N} \Psi\left(|u_i - f_j|^2\right)$$

Minimising $E(\mathbf{u})$

- $E(\mathbf{u})$ can be minimised by gradient descent.
- One can obtain the following iterative formula:

$$u_i^0 = f_i \quad , \qquad u_i^{k+1} = \frac{\sum_{j=1}^{N} g\left(|u_i^k - f_j|^2\right) f_j}{\sum_{j=1}^{N} g\left(|u_i^k - f_j|^2\right)}$$

Local M-smoothers

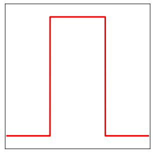
- Estimates the grey value of a pixel from a *local* neighbourhood only.
- Modification: introduction of a weighting term.

$$E_D(\mathbf{u}) = \sum_{i=1}^{N} \sum_{j \in \mathcal{B}(i)} \Psi_D\left(|u_i - f_j|^2\right) w_D\left(|x_i - x_j|^2\right)$$

$\mathcal{B}(i)$ is introduced to make the index j run through the neighbourhood of x_i.

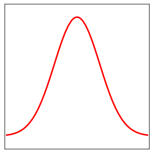
- Examples for weighting functions:
  - hard disk-shaped window

$$w(s^2) = \begin{cases} 1 & s^2 < \theta \\ 0 & \text{otherwise} \end{cases}$$



  - soft window

$$w(s^2) = e^{-s^2/\theta^2}$$



Minimising $E_D(\mathbf{u})$

- Similarly to the minimisation of $E(\mathbf{u})$ in the part about histogram operations, one can state an iterative formula:

$$\begin{aligned} u_i^0 &= f_i \\ u_i^{k+1} &= \frac{\sum_{j \in \mathcal{B}(i)} g\left(|u_i^k - f_j|^2\right) w\left(|x_i - x_j|^2\right) f_j}{\sum_{j \in \mathcal{B}(i)} g\left(|u_i^k - f_j|^2\right) w\left(|x_i - x_j|^2\right)} \end{aligned}$$

  where g is called the *tonal weight* and w the *spatial weight*.

- This procedure is called the *W-estimator*. It is one possibility to obtain a solution to the local M-estimation problem.

- Note:
  One is only interested here in the *steady state* for $k \to \infty$, not in the evolution towards the minimiser.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

Bilateral filtering

- Estimate the image gradient magnitude as a sum of squared differences from a pixel to its neighbours. The discrete smoothness penaliser reads

$$E_S(\mathbf{u}) = \sum_{i=1}^{N} \Psi_S \left(\sum_{j \in \mathcal{N}(i)} |u_i - u_j|^2 \right)$$

  where $\mathcal{N}(i)$ is the set of 4-neighbours of a pixel i.

- Interesting fact: Exchanging the order of summation and penalisation gives an *anisotropic* smoothness measure

$$E_S(\mathbf{u}) = \sum_{i=1}^{N} \sum_{j \in \mathcal{N}(i)} \Psi_S \left(|u_i - u_j|^2 \right)$$

- Increasing the neighbourhood, the smoothness term becomes

$$E_S(\mathbf{u}) = \sum_{i=1}^{N} \sum_{j \in \mathcal{B}(i)} \Psi_S \left(|u_i - u_j|^2 \right) w_S \left(|x_i - x_j|^2 \right)$$

Minimising $E_S(\mathbf{u})$

- The smoothness term can be minimised by an iterative procedure

$$\begin{aligned} u_i^0 &= f_i \\ u_i^{k+1} &= \frac{\sum_{j \in \mathcal{B}(i)} g\left(|u_i^k - u_j^k|^2\right) w\left(|x_i - x_j|^2\right) u_j^k}{\sum_{j \in \mathcal{B}(i)} g\left(|u_i^k - u_j^k|^2\right) w\left(|x_i - x_j|^2\right)} \end{aligned}$$

  This is called *iterative bilateral filter*. This stands in contrast to the original proposal as a heuristic algorithm.

- Note:
  Bilateral filtering uses the *evolving image*. For $k \to \infty$, one would obtain a flat image!

Overview of the iterative formulas

- Histogram operations

$$u_i^{k+1} = \frac{\sum_{j=1}^{N} g\left(|u_i^k - f_j|^2\right) f_j}{\sum_{j=1}^{N} g\left(|u_i^k - f_j|^2\right)}$$

- Local M-smoothers

$$u_i^{k+1} = \frac{\sum_{j \in \mathcal{B}(i)} g\left(|u_i^k - f_j|^2\right) w\left(|x_i - x_j|^2\right) f_j}{\sum_{j \in \mathcal{B}(i)} g\left(|u_i^k - f_j|^2\right) w\left(|x_i - x_j|^2\right)}$$

- Iterative Bilateral filtering

$$u_i^{k+1} = \frac{\sum_{j \in \mathcal{B}(i)} g\left(|u_i^k - u_j^k|^2\right) w\left(|x_i - x_j|^2\right) u_j^k}{\sum_{j \in \mathcal{B}(i)} g\left(|u_i^k - u_j^k|^2\right) w\left(|x_i - x_j|^2\right)}$$

A Unifying Framework

$$E_D(\mathbf{u}) = \sum_{i=1}^{N}\sum_{j=1}^{N} \Psi_D\left(|u_i - f_j|^2\right) w_D\left(|x_i - x_j|^2\right)$$

$$E_S(\mathbf{u}) = \sum_{i=1}^{N}\sum_{j=1}^{N} \Psi_S\left(|u_i - u_j|^2\right) w_S\left(|x_i - x_j|^2\right)$$

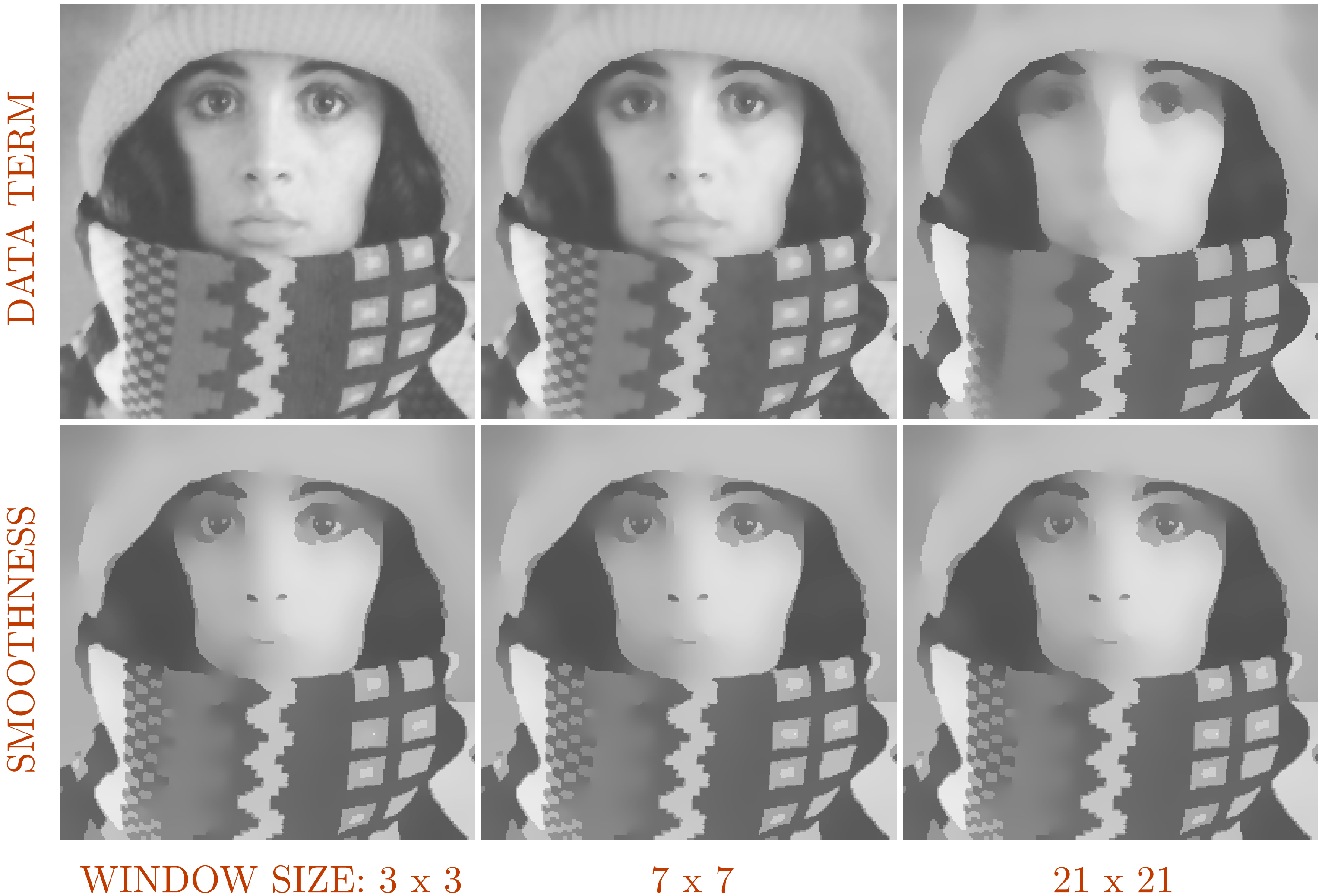
- The unifying energy looks as follows (convex combination):

$$E(\mathbf{u}) = \sigma E_D(\mathbf{u}) + (1-\sigma) E_S(\mathbf{u}) \quad 0 \leq \sigma \leq 1$$

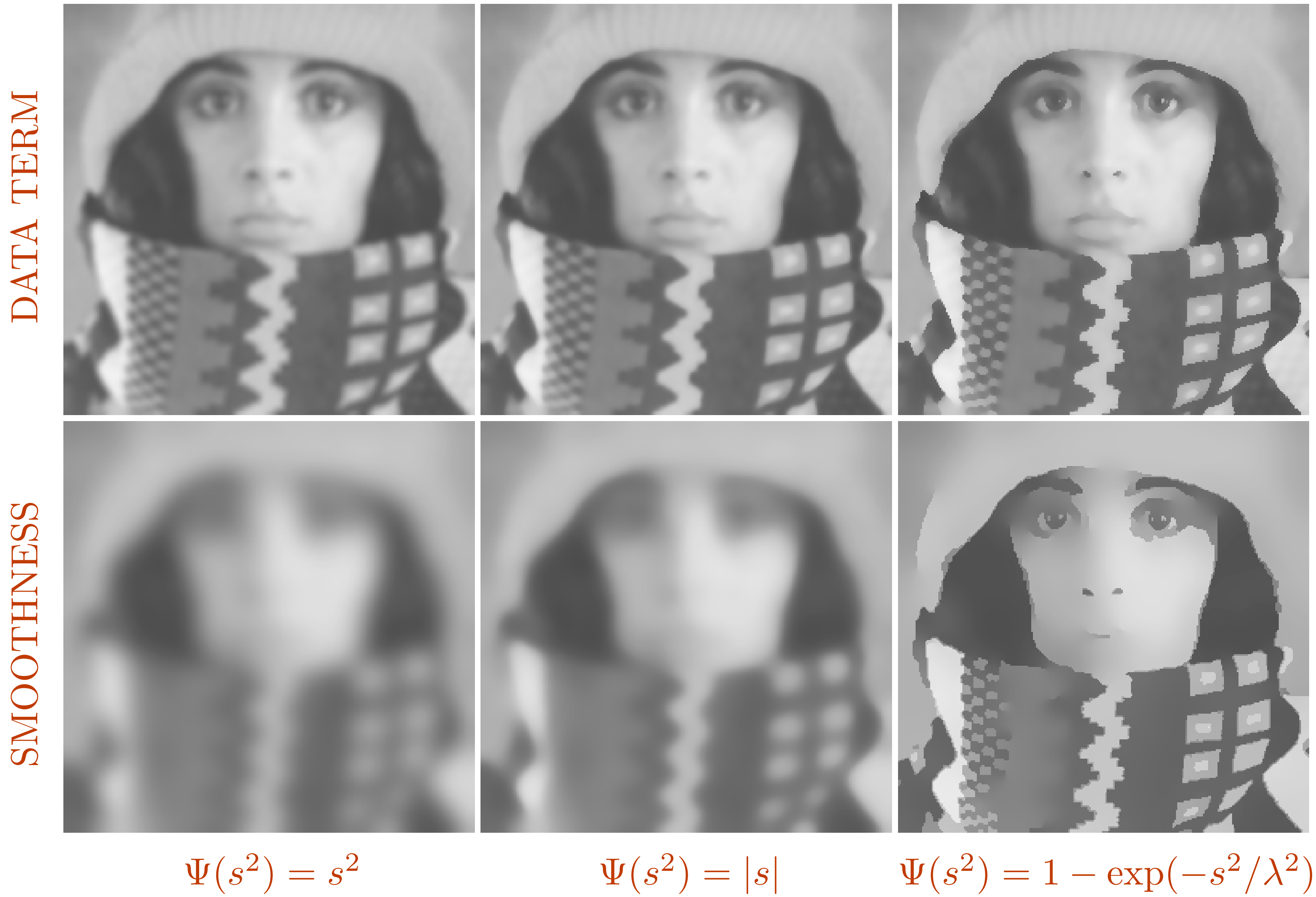
- Advantages: Freedom in selecting
  - the penaliser type Ψ_D and Ψ_S as well as its parameters and
  - the balance between smoothness and data terms.

Experiments

Input images for the filtering examples. **Left:** Noise-free version. **Right:** Noisy version. **Authors:** P. Mrázek, J. Weickert, A. Bruhn.

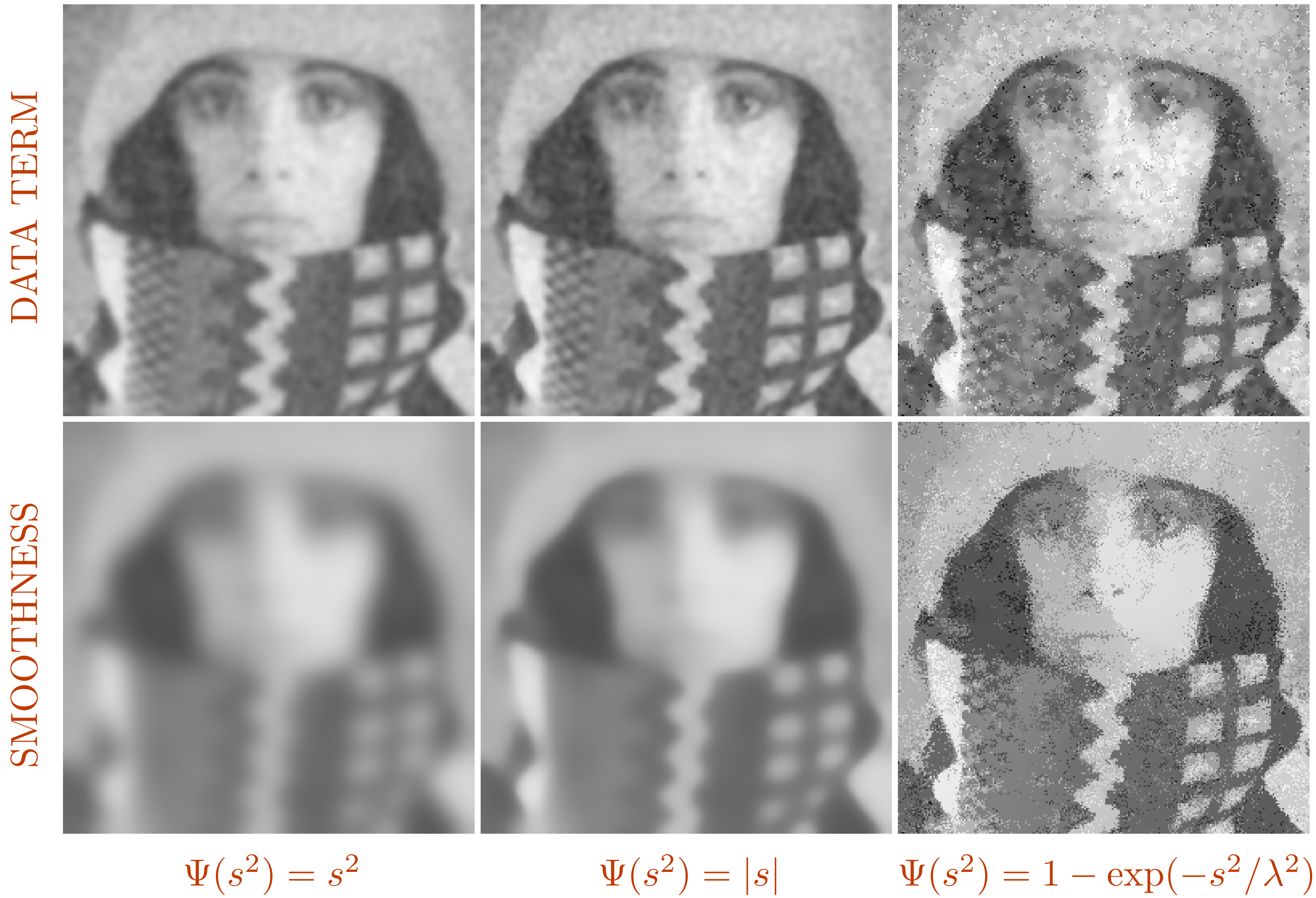


Filtering using the penaliser $\Psi(s^2) = 1 - e^{-s^2/\lambda^2}$ with soft window of varied size of the spatial neighbourhood ($\theta = 1, \theta = 3, \theta = 10$). **Top:** local M-smoothers (data term, steady state after iteration). **Bottom:** bilateral filtering (smoothness term, 200 iterations). **Authors:** P. Mrázek, J. Weickert, A. Bruhn.



Effect of the penaliser type on the filtering result, starting with the noise-free image (with soft spatial weighting $w(s^2) = e^{-s^2/\theta^2}$, $\theta = 3$). **Authors:** P. Mrázek, J. Weickert, A. Bruhn.

Experiments



Effect of the penaliser type on the filtering result, starting from the noisy image (filtering in a soft window $w(s^2) = e^{-s^2/\theta^2}$, $\theta = 3$). **Authors:** P. Mrázek, J. Weickert, A. Bruhn.

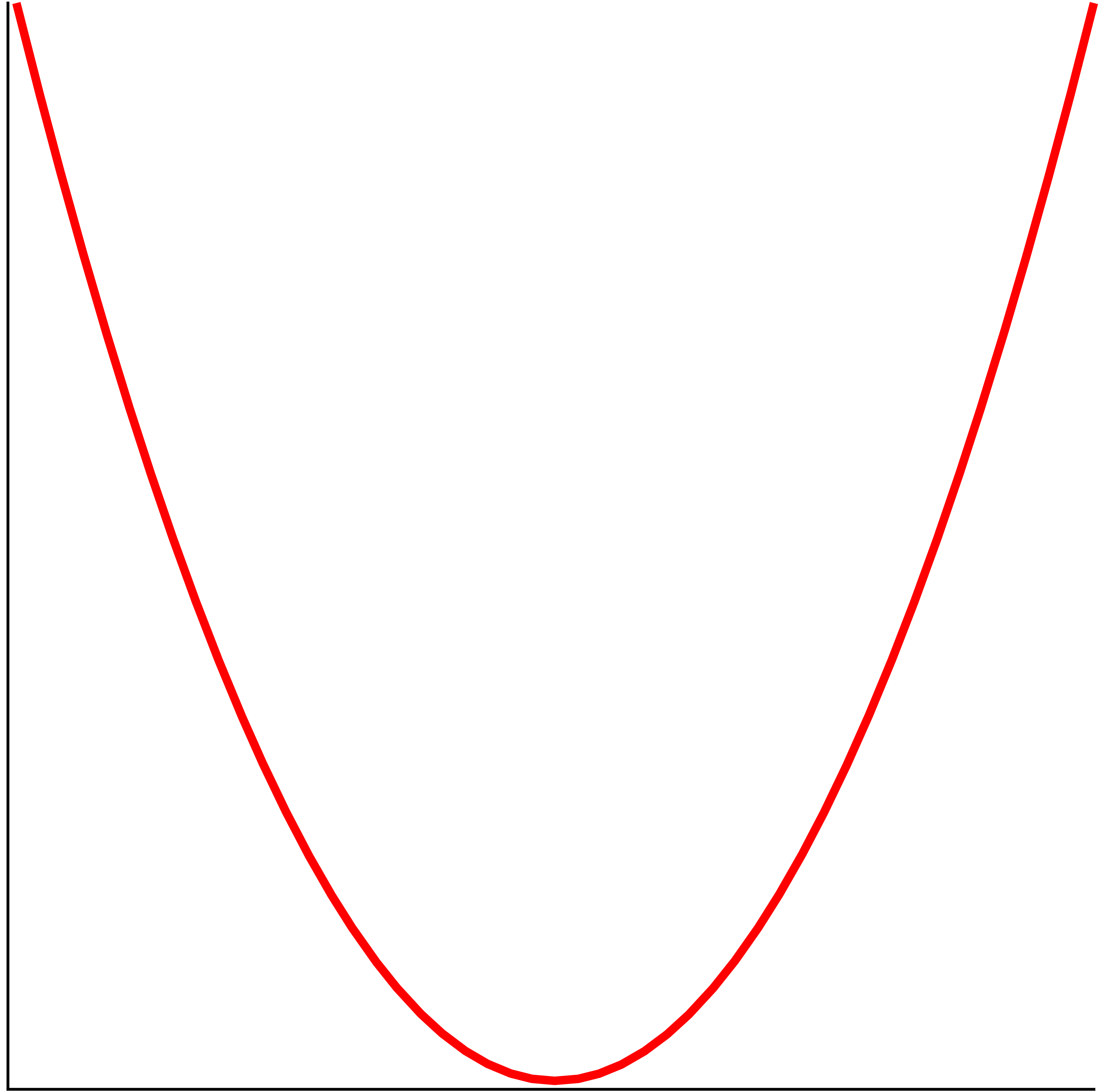
Summary

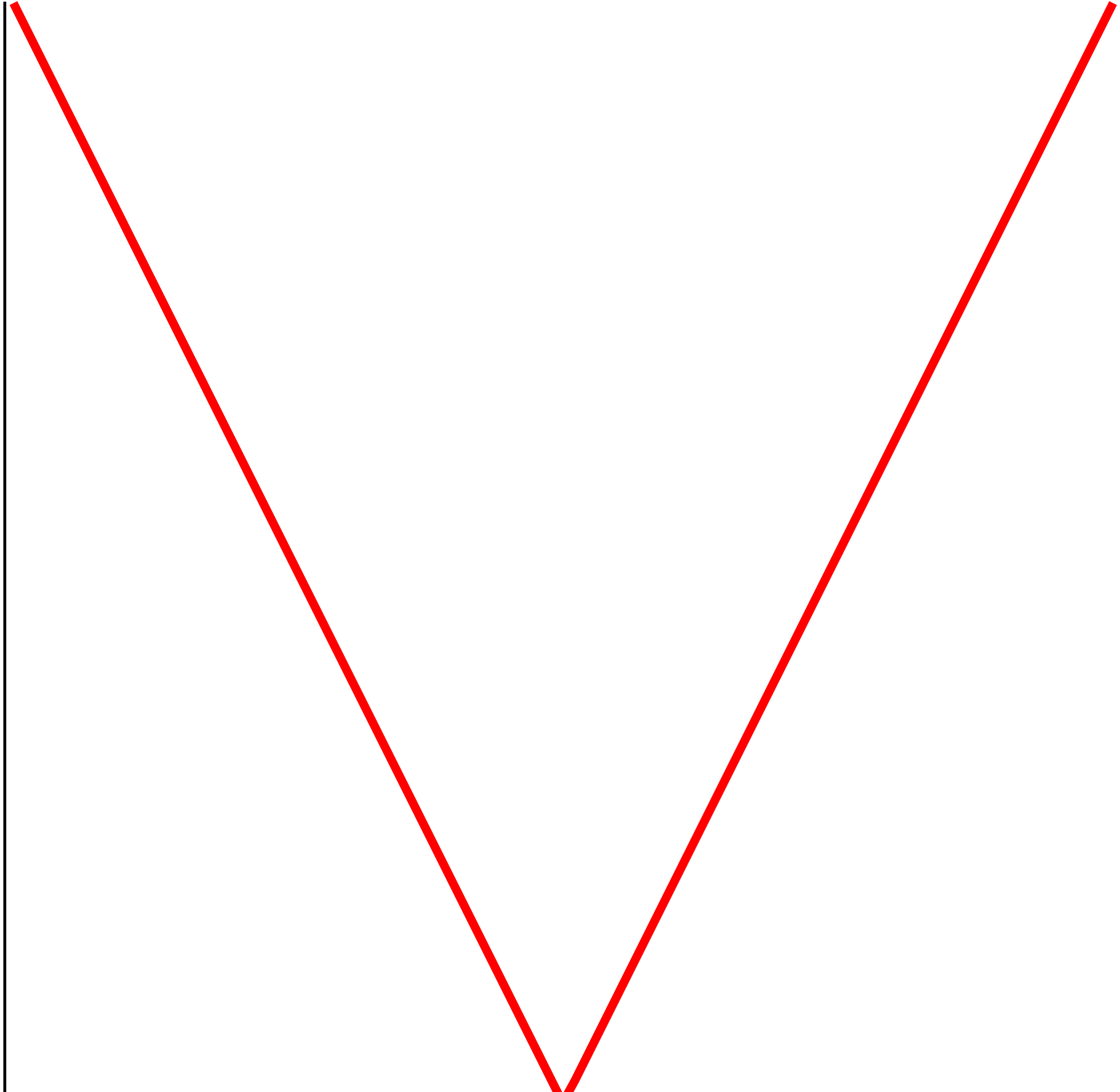
- Local M-smoothers minimise an energy with a windowed *data* term. Performing a fixed-point iteration results in *W-estimators*.
- Iterative bilateral filtering minimises an energy with a windowed *smoothness* term.
- The unifying framework minimises functionals, combining nonlocal data term and nonlocal smoothness term.
- It helps for a better understanding of the different methods and allows to combine the advantages of known filters to make way for new techniques.
- Filters are for example
  - Local M-smoothers,
  - Bilateral filters,
  - and more.

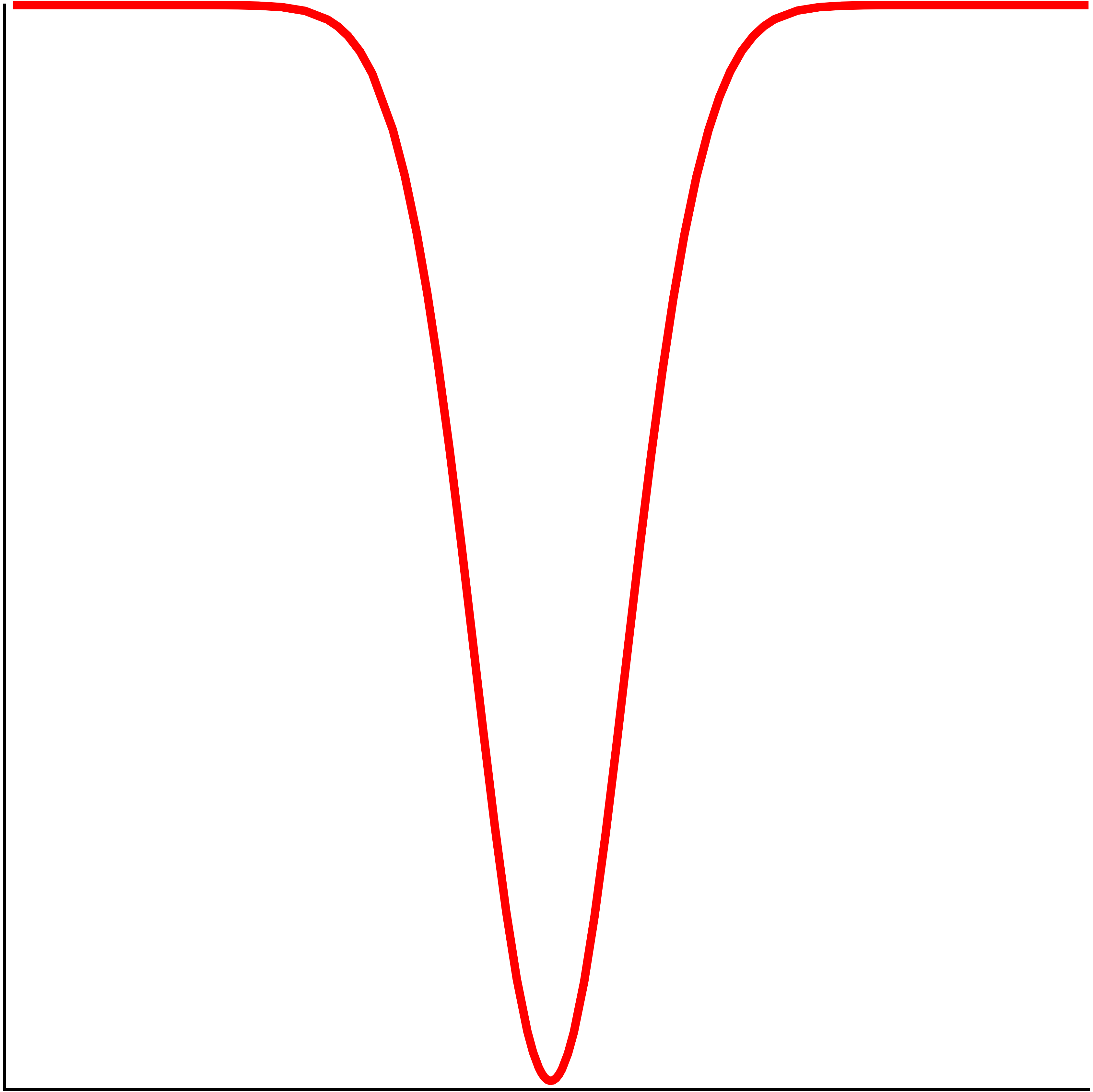
References

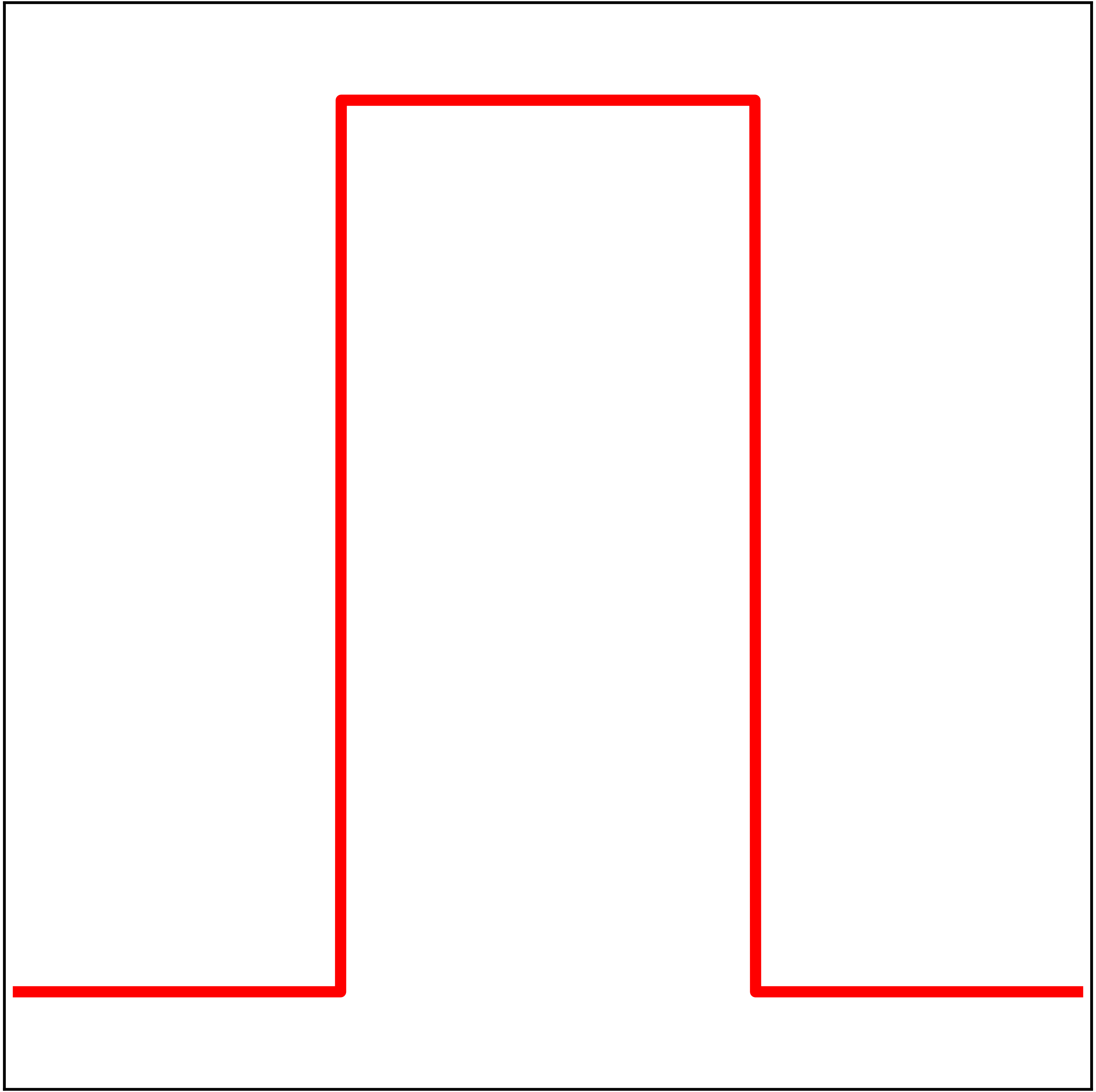
- R. Klette, R. Kozera, L. Naakes, J. Weickert (Eds.): Geometric Properties for Incomplete Data, 335 – 352. Springer Dordrecht, 2006.
  (unified denoising model)
- Lecture "Differential Equations in Image Processing and Computer Vision". SS 2008. `http://www.mia.uni-saarland.de/Teaching/dic08.shtml`.
  (Lecture 25 gives insights in the unifying framework)
- C. K. Chu, I. Glad, F. Godtliebsen, J. S. Marron: Edge-preserving smoothers for image processing. *Journal of the American Statistical Association*, Vol. 93, No. 442, 526–556, 1998.
  (local M-smoothers)
- G. Winkler, V. Aurich, K. Hahn, A. Martin: Noise reduction in images: some recent edge-preserving methods. *Pattern Recognition and Image Analysis*, Vol. 9, No. 4, 749–766, 1999.
  (W-estimators)
- C. Tomasi, R. Manduchi: Bilateral filtering for gray and color images. *Proc. Sixth International Conference on Computer Vision* (Bombay, India, Jan. 1998), 839–846, Narosa, 1998.
  (bilateral filter)

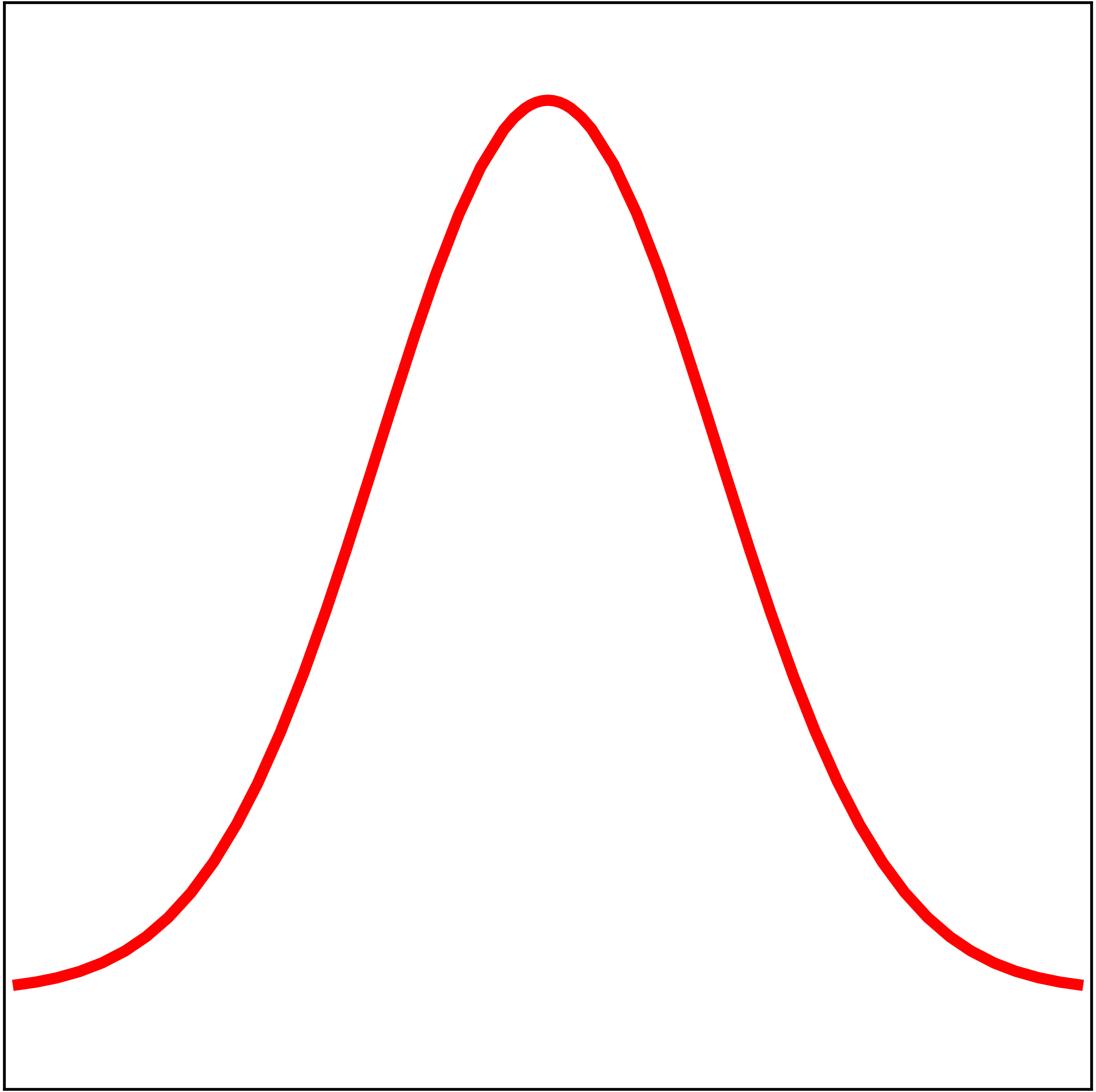
Thank you for your attention!

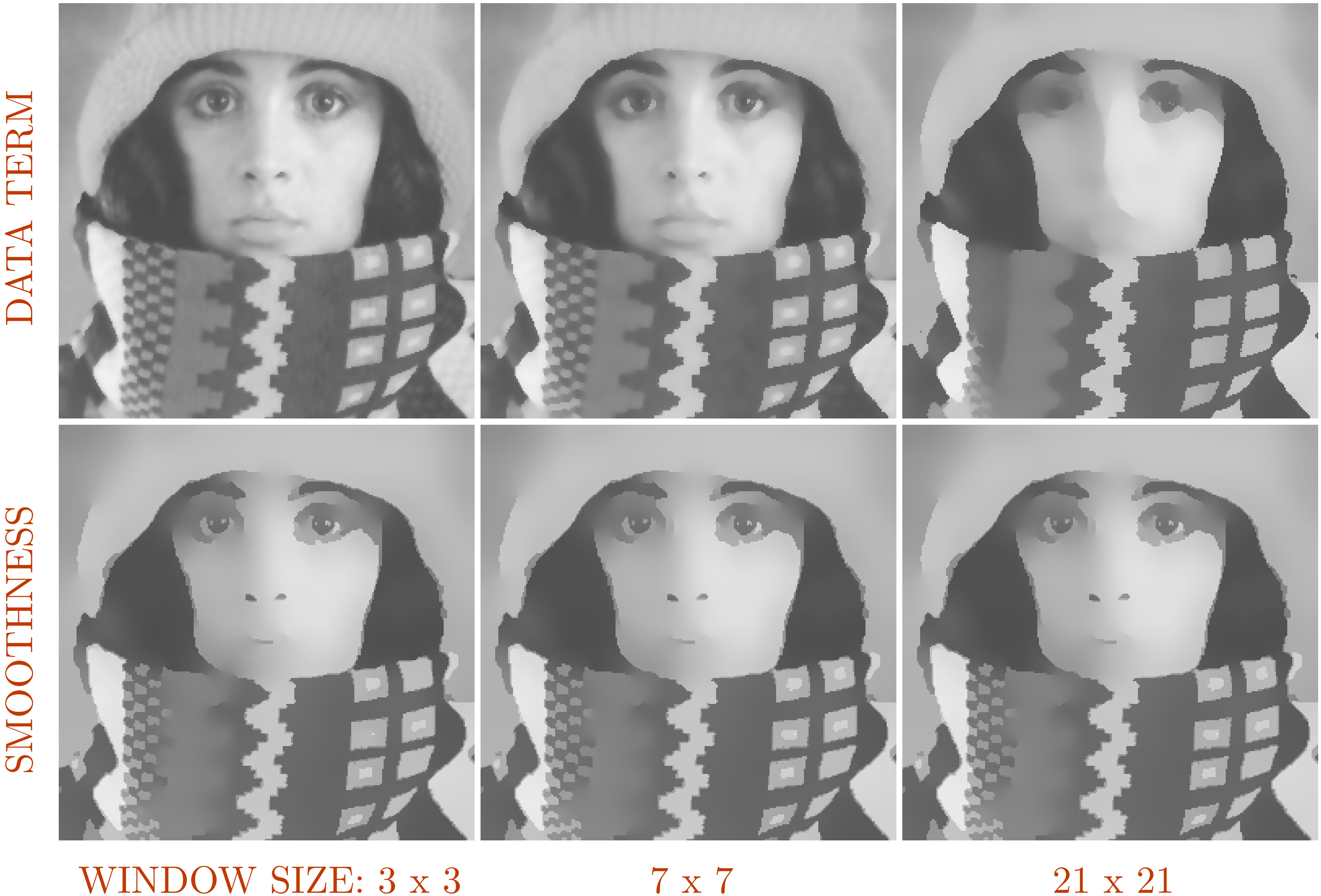


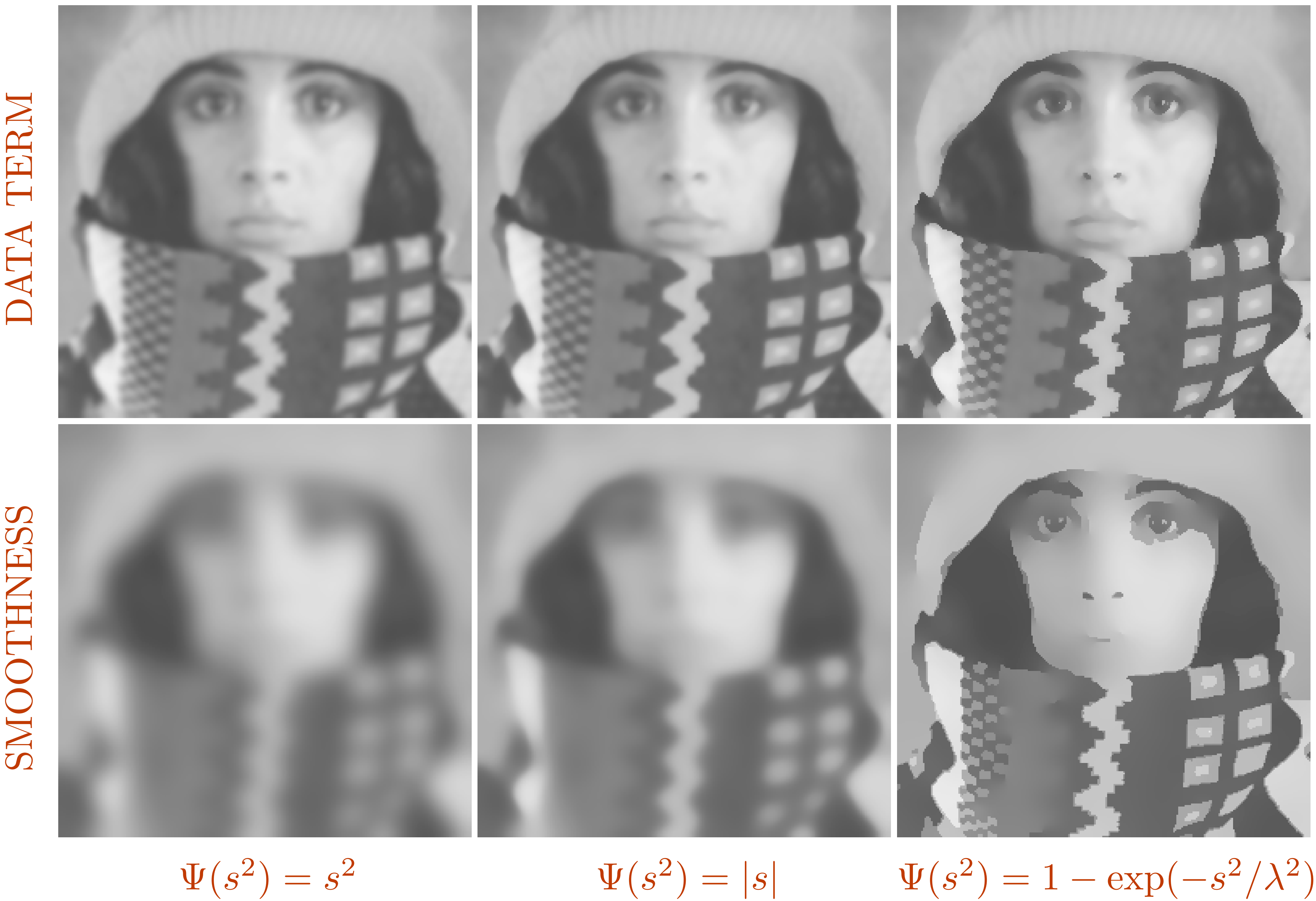


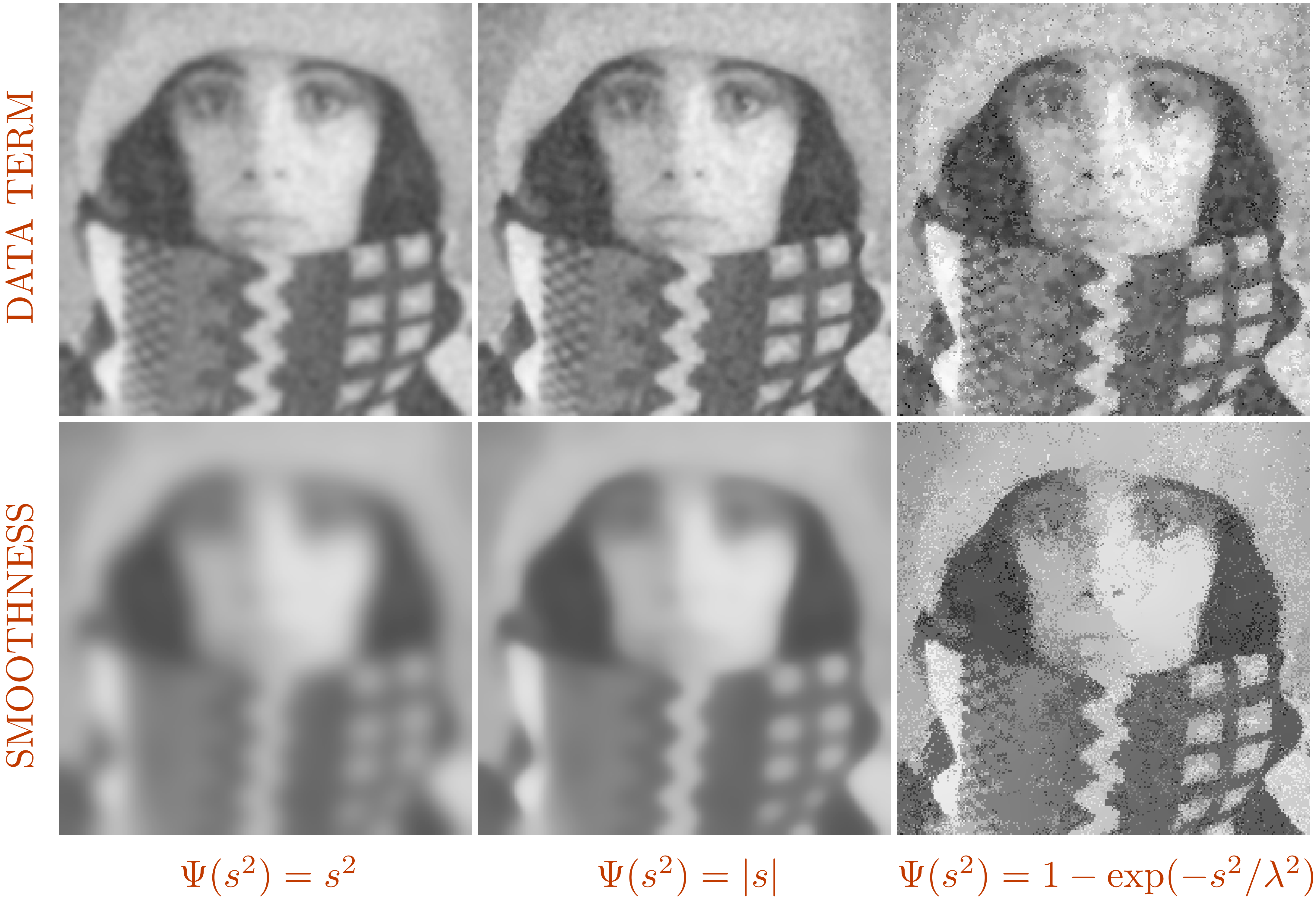








DATA TERM
SMOOTHNESS
WINDOW SIZE: 3 x 3
7 x 7
21 x 21


DATA TERM
SMOOTHNESS
$\Psi(s^2) = s^2$
$\Psi(s^2) = |s|$
$\Psi(s^2) = 1 - \exp(-s^2/\lambda^2)$


DATA TERM
SMOOTHNESS
$\Psi(s^2) = s^2$
$\Psi(s^2) = |s|$
$\Psi(s^2) = 1 - \exp(-s^2/\lambda^2)$