Lecture 23:

Curvature-Based Morphology II: Affine Morphological Scale-Space

Contents

Motivation

- In Lecture 22, we have considered the mean curvature motion

$$\partial_t u \;=\; |\nabla u| \, \mathrm{curv}(u).$$

- The corresponding curve evolution could be written as

$$\partial_t \mathbf{c}(p,t) \;=\; \partial_{\nu\nu} \mathbf{c}(p,t)$$

  where ν is the Euclidean invariant arc-length.

- Is it possible to find related evolutions that are not only invariant under Euclidean transformations (rotations, translations), but also under affine ones ?

Affine Morphological Scale-Space

Affine Transformations:

- given by
$$\mathbf{x} \;\mapsto\; A\mathbf{x} + \mathbf{b}$$
where $\mathbf{b} \in \mathbb{R}^2$ is a translation vector and $A \in \mathbb{R}^{2\times 2}$ is invertible.
- arise as shape distortions of planar objects when being observed from a large distance under different angles

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

Affine Invariant Intrinsic Diffusion

(Sapiro/Tannenbaum 1993)

- Idea: replace Euclidean arc-length v in intrinsic heat flow
$$\partial_t \mathbf{c}(p,t) \;=\; \partial_{vv}\,\mathbf{c}(p,t)$$
by an *"affine invariant arc-length"* (Blaschke 1923)
$$s(p,t) \;:=\; \int_0^p \Big(\det\big(\mathbf{c}_\rho(\rho,t),\, \mathbf{c}_{\rho\rho}(\rho,t)\big)\Big)^{\frac{1}{3}} d\rho.$$
- By virtue of
$$\partial_{ss}\mathbf{c}(p,t) \;=\; \big(\kappa(p,t)\big)^{\frac{1}{3}}\,\mathbf{n}(p,t)$$
one gets the *affine invariant heat flow (affine shortening flow)*
$$\partial_t\mathbf{c}(p,t) \;=\; \big(\kappa(p,t)\big)^{\frac{1}{3}}\,\mathbf{n}(p,t).$$

Affine Morphological Scale-Space (3)

Corresponding Affine Invariant Image Evolution

(Alvarez/Guichard/Lions/Morel 1993)

- follows the PDE

$$\partial_t u \;=\; |\nabla u| \left(\mathrm{curv}(u)\right)^{\frac{1}{3}}$$

- can be rewritten as

$$\begin{aligned}\partial_t u \;&=\; \left(u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}\right)^{\frac{1}{3}} \\ &=\; |\nabla u|^{\frac{2}{3}}\, u_{\xi\xi}^{\frac{1}{3}}\end{aligned}$$

- This evolution is called *affine morphological scale-space (AMSS)*.
- Alvarez et al. showed in an axiomatic way that it is the unique affine invariant scale-space.
- Since it has more invariances than other scale-spaces, they named it *fundamental equation in image processing*.

Affine Morphological Scale-Space (4)

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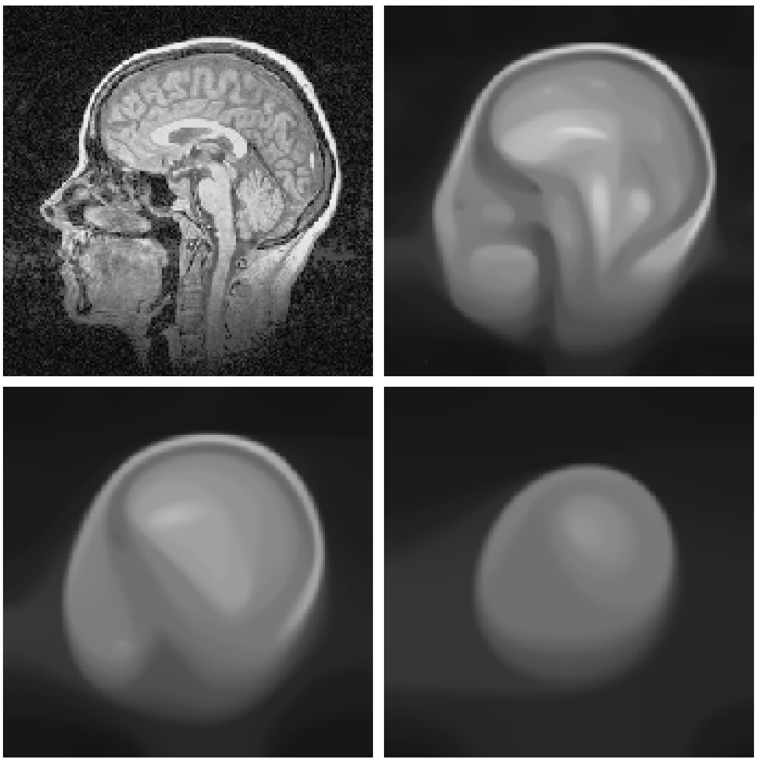

Affine morphological scale-space. (a) **Top left:** Original image, $\Omega = (0, 236)^2$. (b) **Top right:** $t = 20$. (c) **Bottom left:** $t = 50$. (d) **Bottom right:** $t = 140$.

Theoretical Results

basically same as for MCM, with the following exceptions:

- Closed curves shrink to *elliptical points*.
- Ellipses maintain their eccentricity during shrinkage.
- time for shrinking a circle of radius σ (or ellipse of area $\pi\sigma^2$) to a point:

$$T = \tfrac{3}{4}\sigma^{\frac{4}{3}}$$

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Numerical Scheme

- Typically one uses the equation

$$u_t \;=\; \left(u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}\right)^{\frac{1}{3}}$$

  replaces u_t by forward differences, and the spatial derivatives by central differences.

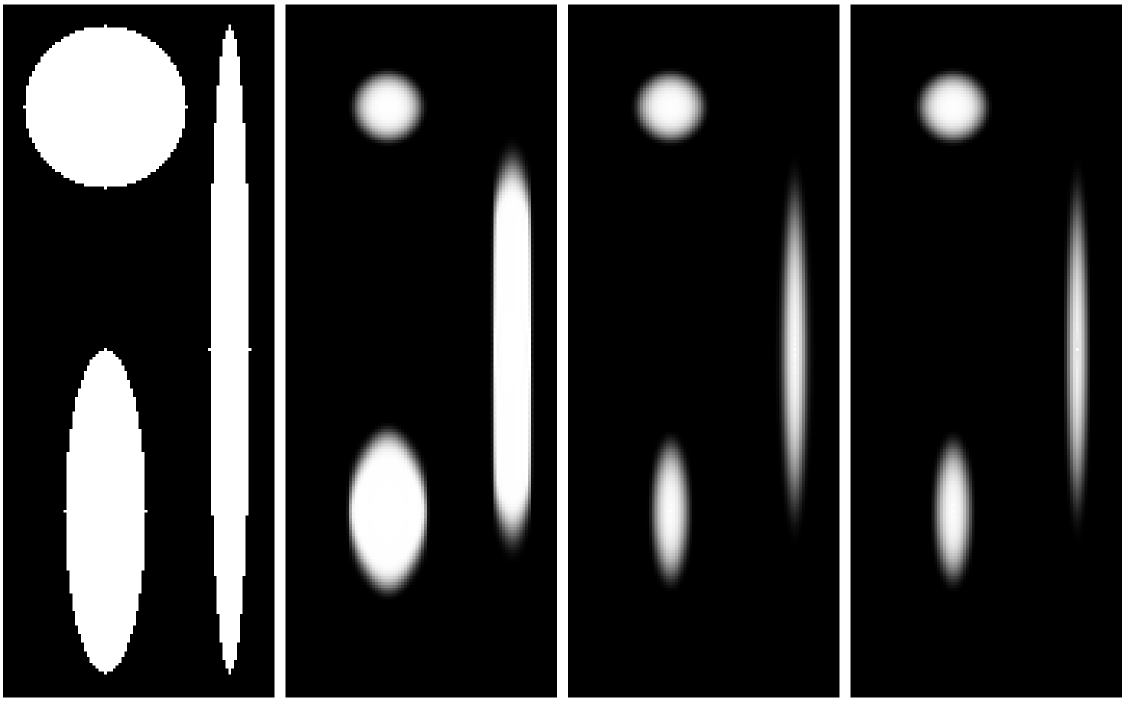
- no stability theory available for this explicit scheme;
  can violate extremum principle
- typical step size for experimental stability ($h_1 = h_2 = 1$):

$$\tau \;\leq\; 0.1.$$

- However, a good approximation of affine invariance may require much smaller time steps, for instance

$$\tau \;\leq\; 0.01.$$

Numerical Scheme (2)



Test of affine invariance of an explicit scheme for AMSS. (a) **Left:** Original image with four ellipses of equal area, $\Omega = (0,94) \times (0,240)$. Under AMSS, these ellipses should keep their eccentricity. (b) **Middle left:** Result with 500 iterations of time step size $\tau = 0.1$. (c) **Middle right:** 5000 iterations with $\tau = 0.01$. (d) **Right:** 50000 iterations with $\tau = 0.001$.

Extensions

Extensions

Three-Dimensional AMSS

(Caselles/Sbert 1996)

- is given by the equation

$$\partial_t u \;=\; |\nabla u|\, \mathrm{sgn}(\kappa_1)\, (\kappa_1 \kappa_2)^{1/4}$$

(κ_1, κ_2: principal curvatures of isosurface)

- In contrast to 2-D, topological changes may occur
(reminescent of diffusion filters when going from 1-D to 2-D)

AMSS for Movies

(Guichard 1998)

- There exists also a unique morphological, affine, and Galilean invariant scale-space for movies.
- can also be justified in an axiomatic way

Application: Corner Detection

(Alvarez/Morales 1997)

Problems

- Corner detection is sensitive to noise.
- Embedding into Gaussian scale-space increases robustness, but dislocates corners.
- Corners cannot always be traced back in scale (Sporring et al. 2000).

Goal

- embedding of corner detection into a better suited scale-space with a well-founded downfocusing strategy

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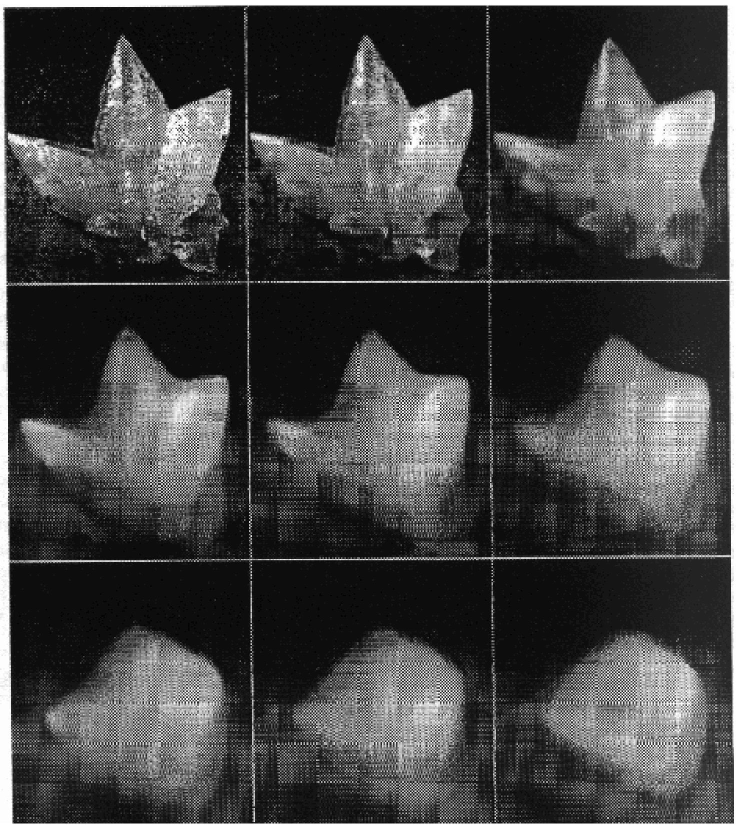

Evolution of a crystal under AMSS. **Authors:** L. Alvarez and F. Morales (1997).

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Method

- create a stack of images in AMSS
- identify corners as curvature extrema of isophotes
- start with significant corners at a coarse scale
- analytical (!) results from the theory of AMSS (p. 15):
  – corners move along a line
  – speed is a simple function of corner angle
  allow to extrapolate corner location and angle to scale $t = 0$

Why AMSS ?

- curvature-based morphological scale-spaces cannot create new corners:
  number of curvature extrema is nonincreasing
- affine invariance guarantees that:
  all corners evolve in the same manner (but with different speed), because different angles can be transformed into each other by affine transformations

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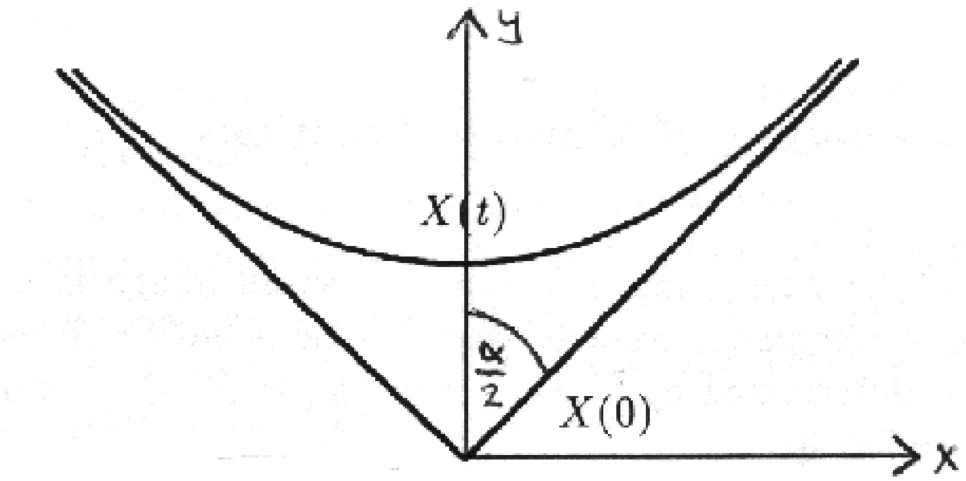


Under AMSS, a corner becomes a hyperbola. **Authors:** L. Alvarez and F. Morales (1997).

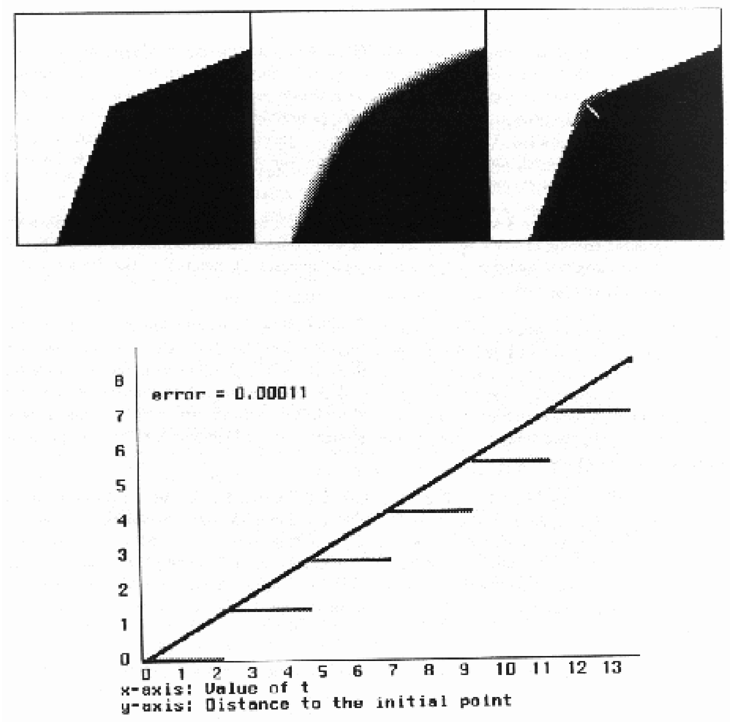
MIA
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

Corner evolution under AMSS

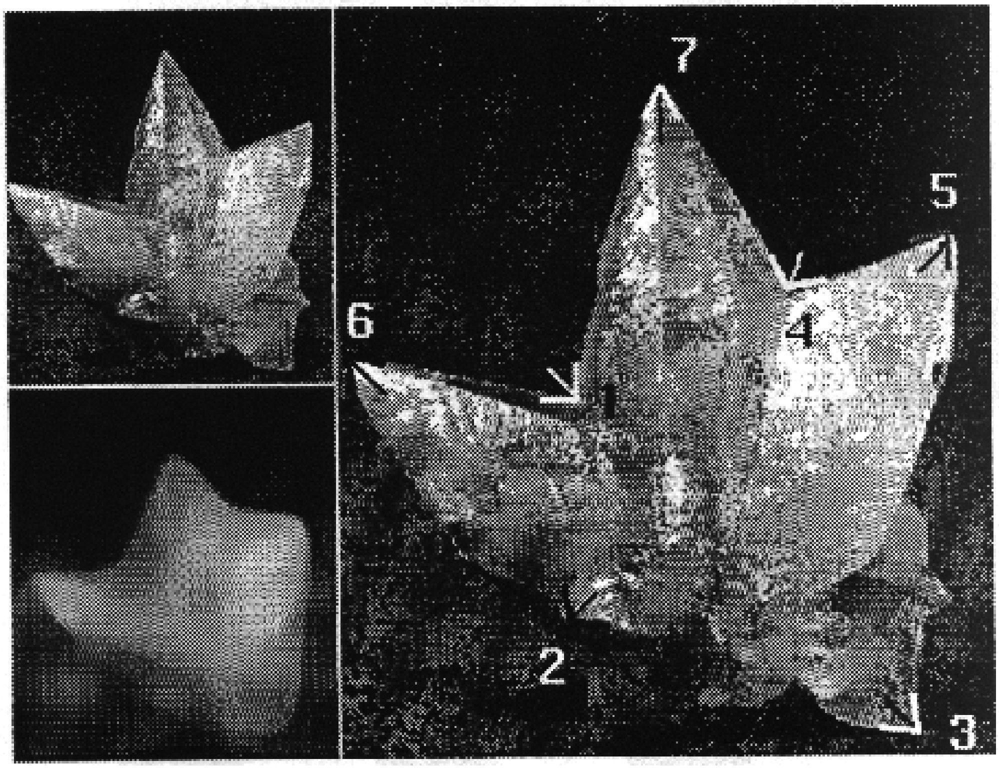
- Evolution of the previous corner with angle α under the (temporally rescaled) AMSS
$$u_t \;=\; t^{1/3}\left(u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}\right)^{1/3}$$
is analytically given by the hyperbola
$$y \;=\; \lambda\sqrt{t^2 + \lambda^2 x^2}$$
with $\lambda := 1/\sqrt{\tan\frac{\alpha}{2}}$.

- Corners at $x = 0$ move linearly in time
$$y = \lambda t$$
where the speed $\lambda = 1/\sqrt{\tan\frac{\alpha}{2}}$ depends on the corner angle α.

MIA
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21



(a) **Top:** From left to right: original corner, smoothed by AMSS, and extrapolated to scale $t = 0$. (b) **Bottom:** Under AMSS, the corner displacement is a linear function in time. This allows extrapolation to the initial location at $t = 0$. **Authors:** L. Alvarez and F. Morales (1997).



(a) **Left:** Crystal image and AMSS-smoothed version. (b) **Right:** The seven most significant corners after linear extrapolation. **Authors:** L. Alvarez and F. Morales (1997).

Summary

- Affine morphological scale-space (AMSS) is the affine invariant analogue to mean curvatute motion (MCM).
- Level lines follow an intrinsic heat equation where the Euclidean arc length is replaced by an affine invariant one.
- The corresponding image evolution can be derived in an axiomatic way.
- similar theoretical properties as MCM
- numerically difficult, if good approximations to affine invariance are required
- AMSS for corner detection:
  - identifies corners at coarse scale (as curvature maxima)
  - analytical formula for stable extrapolation to $t = 0$
  - AMSS guarantees that all corners evolve in the same way

References

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  (This lecture is based on Section 1.6.)
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  (axiomatic derivation of AMSS as image evolution)
- G. Sapiro, A. Tannenbaum: Affine invariant scale-space. *International Journal of Computer Vision*, Vol. 11, 25–44, 1993.
  (AMSS as curve evolution)
- V. Caselles, C. Sbert: What is the best causal scale-space for 3-D images? *SIAM Journal on Applied Mathematics*, Vol. 56, 1199–1246, 1996.
  (3-D versions of MCM and AMSS)
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Assignment C6 – Classroom Work

Problem 1 (Continuous-Scale Morphology Morphology)

In Lecture 20, we have seen that dilation with disc-shaped structuring elements of radius t can be understood as curve evolution

$$\partial_t \mathbf{c} \;=\; \mathbf{n}$$

or as image evolution

$$\partial_t u \;=\; \nabla u \;.$$

Determine the evolution equations for dilation with a family of structuring elements whose shape is given as an ellipse.

For simplicity, you can first start with an ellipse alligned to the coordinate system and the half-axes a and b. More general ellipses can be obtained by rotation.

Is there a more elegant and general way to express the ellipses and the evolution?

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Problem 2 (Numerical Schemes for Continuous-Scale Morphology)

We deal with the scheme of Osher and Sethian for continuous-scale morphology. Consider the 1-D case with $h = 1.0$, $\tau = 0.5$ and the following initial data:

$$f_i = \begin{cases} 1 & : \quad i \leq 0\,, \\ 0 & : \quad \text{else.} \end{cases}$$

Consider also a set-based scheme for dilation/erosion. Compare and discuss the results of the following procedures.

(a) With the set-based scheme. Perform exactly one dilation step.

(b) With the set-based scheme. Perform (i) at first one dilation step followed by (ii) one erosion step.

(c) With the Osher-Sethian scheme. Perform two dilation steps.

(d) With the Osher-Sethian scheme. Perform (i) at first one dilation step followed by (ii) one erosion step.