Lecture 22:

Curvature-Based Morphology I: Mean Curvature Motion

Contents

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Motivation

- So far we have only considered morphological evolution of type

$$\partial_t u = \pm |\nabla u|.$$

- They correspond to the classical morphological processes of dilation / erosion with a disc of radius t.
- They propagate level curves (isophotes) with constant speed in outer / inner normal direction

$$\partial_t \mathbf{c} = \pm \mathbf{n}.$$

- However, the level set framework in Lecture 20 explicitly allows for curve evolutions of type

$$\partial_t \mathbf{c} = \beta(\kappa)\, \mathbf{n}.$$

with a curvature-dependent speed function $\beta(\kappa)$.
- Do such curve evolutions and their corresponding image evolutions make sense ?

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Mean Curvature Motion as Image Evolution

From Homogeneous Diffusion to Mean Curvature Motion

- The Laplacian within homogeneous diffusion can be decomposed into two orthogonal directions:

$$\begin{aligned} \partial_t u &= \Delta u \\ &= \partial_{xx} u + \partial_{yy} u \\ &= \partial_{\eta\eta} u + \partial_{\xi\xi} u, \end{aligned}$$

  where $\eta \parallel \nabla u$ and $\xi \perp \nabla u$.

- $\partial_{\eta\eta} u$ smoothes along *flowlines*, while $\partial_{\xi\xi} u$ smoothes along *isophotes*.

- A PDE that performs only anisotropic smoothing along isophotes is called *(mean) curvature motion (MCM)*:

$$\partial_t u = \partial_{\xi\xi} u$$

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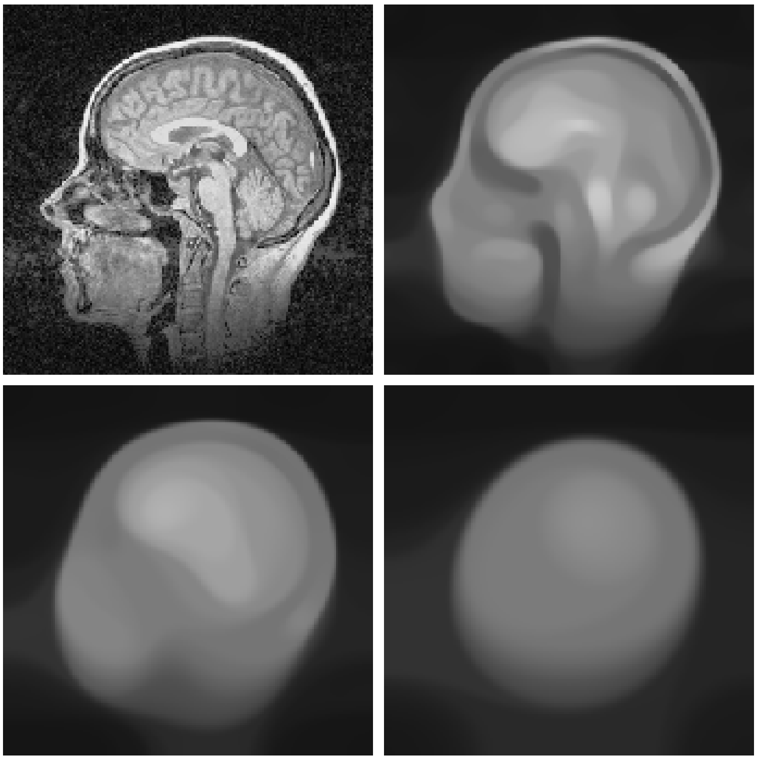

Mean curvature motion. (a) **Top left:** Original image, $\Omega = (0, 236)^2$. (b) **Top right:** $t = 70$. (c) **Bottom left:** $t = 275$. (d) **Bottom right:** $t = 1275$.

Mean Curvature Motion as Image Evolution (3)

Equivalent Notations

- MCM can be written in a number of equivalent ways:

$$
\begin{aligned}
\partial_t u &= \partial_{\xi\xi} u \\
&= \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2} \\
&= \Delta u - \frac{1}{|\nabla u|^2} \nabla u^\top \mathrm{Hess}(u) \nabla u \\
&= |\nabla u| \,\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) \\
&= |\nabla u| \,\mathrm{curv}(u).
\end{aligned}
$$

  where $\mathrm{curv}(u)$ denotes the curvature of an isophote.

Mean Curvature Motion as Image Evolution (4)

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Remarks

- Note that MCM cannot be written in divergence form.
  Unlike diffusion filtering, it does not preserve the average grey value.
- Since it smoothes only along level lines, it is invariant under monotone greyscale rescalings. In this sense it is a morphological process.
- Depending on the sign of the curvature $\mathrm{curv}(u)$, the MCM equation

$$\partial_t u = \mathrm{curv}(u)\,|\nabla u|$$

  may act locally like the dilation $\partial_t u = |\nabla u|$ or the erosion $\partial_t u = -|\nabla u|$.
- However, MCM it a second-order parabolic (diffusion-like) PDE, not a first-order hyperbolic one:
  It has strong regularising properties in forward time direction, and is unstable in backward time direction.
- Note also the difference to shock filtering (Lecture 21):

$$\partial_t u = -\mathrm{sgn}(\Delta u)\,|\nabla u|.$$

Corresponding Curve Evolution

- Let us consider some isophote $\mathbf{c}_0(p)$ of the initial image f, where p denotes the parametrisation.
- Its evolution under the MCM equation $\partial_t u = \mathrm{curv}(u)|\nabla u|$ is given by

$$\begin{aligned} \partial_t \mathbf{c}(p,t) &= \kappa(p,t) \cdot \mathbf{n}(p,t), \\ \mathbf{c}(p,0) &= \mathbf{c}_0(p). \end{aligned}$$

- called *geometric heat equation* or *Euclidean shortening flow* (explanations for this name will follow).
- propagates isophotes in normal direction
- speed of propagation given by their curvature

$$\kappa = \frac{\det(\mathbf{c}_p, \mathbf{c}_{pp})}{|\mathbf{c}_p|^3}.$$

Interpretation as Intrinsic Heat Flow

- The *Euclidean arc-length* of a curve $\mathbf{c}(p,t)$,

$$v(p,t) := \int_0^p |\mathbf{c}_\rho(\rho,t)| \, d\rho$$

is invariant under Euclidean transformations (rotations, translations).

- Because of the differential geometric relation

$$\kappa(p,t) \cdot \mathbf{n}(p,t) = \partial_{vv}\mathbf{c}(p,t)$$

one can regard MCM as Euclidean invariant diffusion of isophotes:

$$\partial_t \mathbf{c}(p,t) = \partial_{vv}\mathbf{c}(p,t)$$

- This explains the name *geometric heat equation*.

Theoretical Results

Theoretical Results for the Curve Evolution

(Huisken 1984, Gage/Hamilton 1986,...):

- smooth solution for some finite interval $[0, T)$
- The number of extrema and inflection points of curvature is nonincreasing.
- Convex curves remain convex.
- Nonconvex ones becomes convex.
- A curve shrinks to a *circular point* for $t \to T$.
- time for shrinking a circle of radius σ to a point:

$$T = \frac{1}{2}\sigma^2.$$

- Shape inclusion principle:
  If one curve lies inside another, they will never intersect during their evolution.
- Thus, all isophotes within a circle of radius σ are removed at time $T = \frac{1}{2}\sigma^2$.

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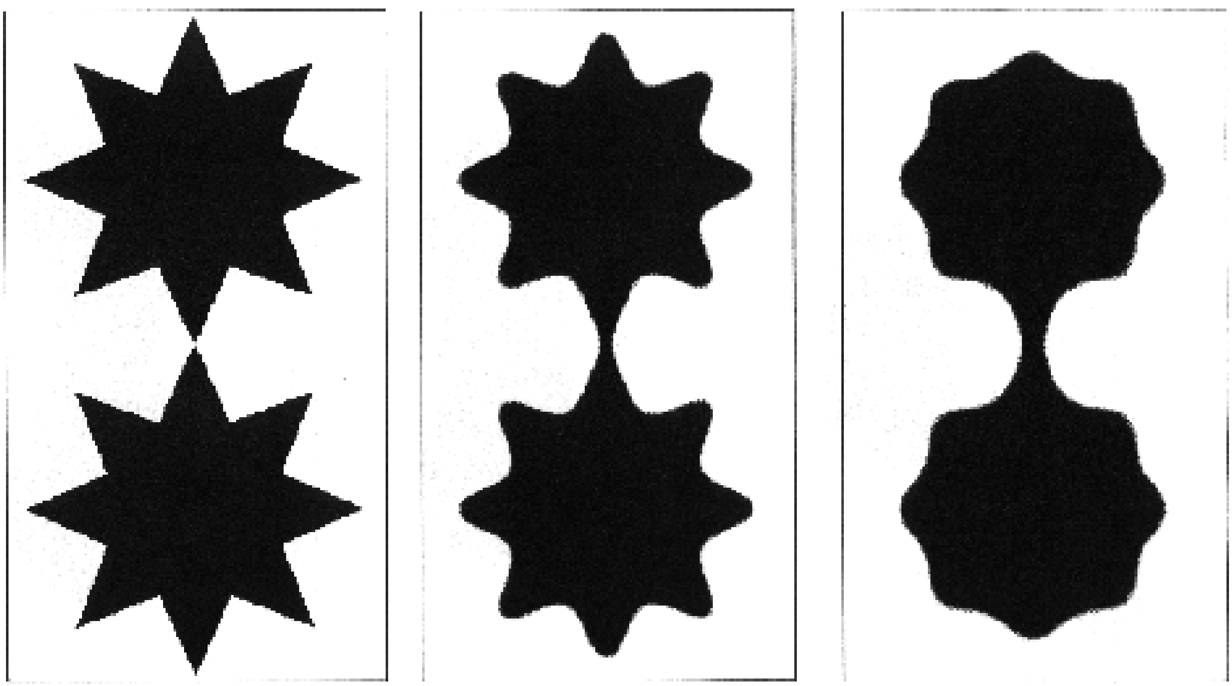
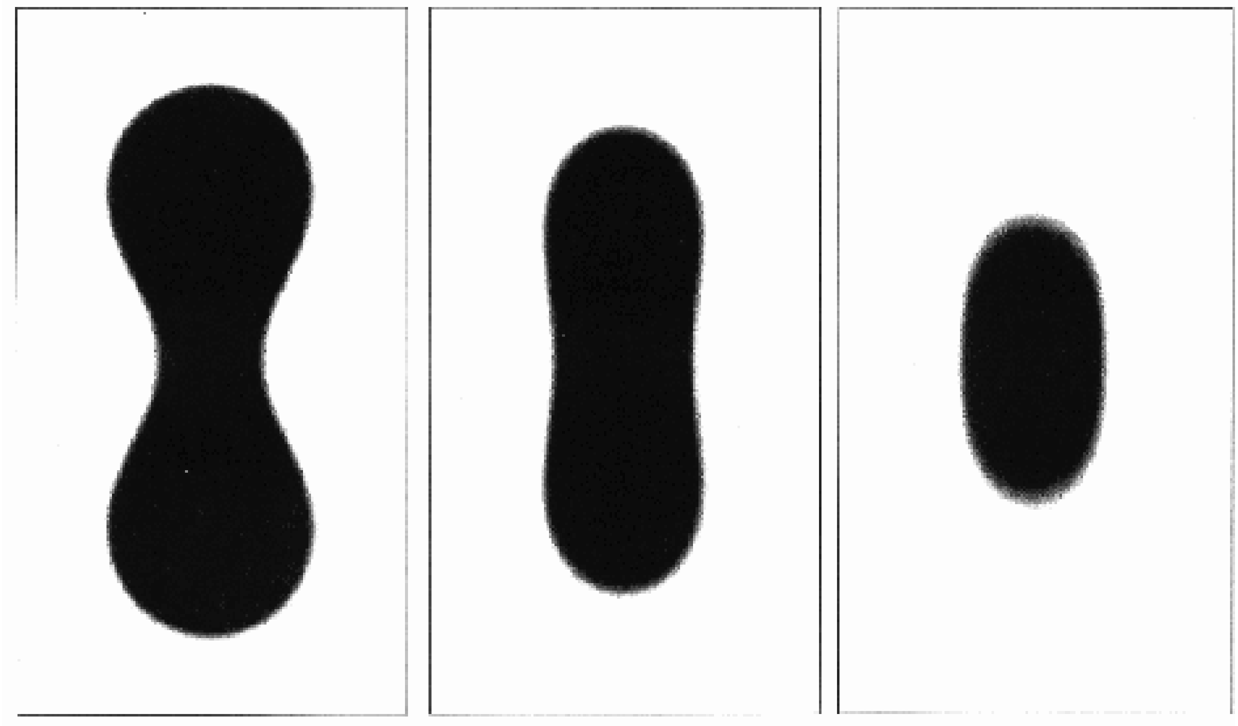


Left to right: Topologically connected objects remain connected under MCM, Continuation see next page. **Author:** H.-W. Kafitz (1996).

Theoretical Results (3)



Left to right: Nonconvex objects become convex under MCM and shrink to circular points. Continuation from the previous page. **Author:** H.-W. Kafitz (1996).

Theoretical Results (4)

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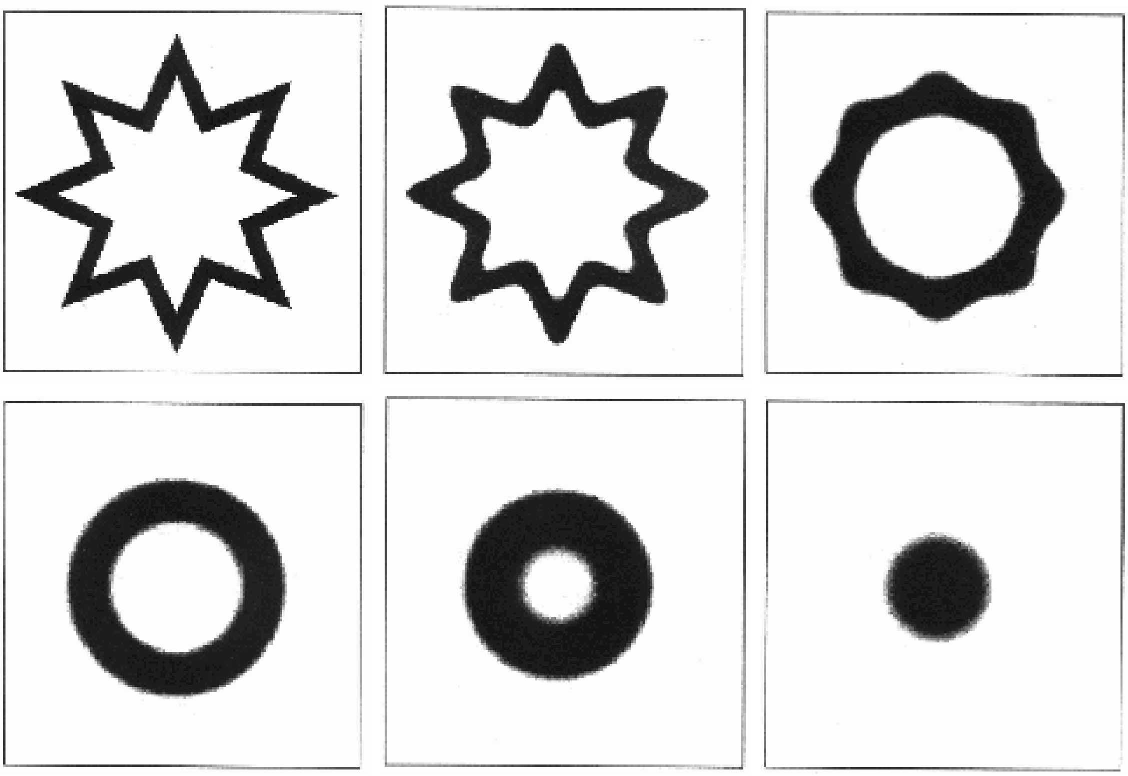


Left to right, top to bottom: Shape inclusion principle: The order of objects within a shape is preserved under MCM. **Author:** H.-W. Kafitz (1996).

Theoretical Results for the Image Evolution

(Alvarez/Guichard/Lions/Morel 1993)

- well-posedness in viscosity sense
- extremum principle, L^∞ stability
- Connected level sets remain connected.
- Unlike diffusion filters, MCM does not create extrema in 2-D.
- morphological technique $\Longrightarrow$ no contrast enhancement

Numerical Scheme

Explicit Scheme

- Typically one uses the equation

$$u_t \;=\; \frac{u_y^2 u_{xx} - 2 u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2},$$

  replaces u_t by forward differences, and the spatial derivatives by central differences.

- no stability theory available for this explicit scheme:
  can violate extremum principle

- typical step size for experimental stability ($h_1 = h_2 = 1$):

$$\tau \;\leq\; 0.25.$$

Connections to Median Filtering

(Guichard/Morel 1997)

- One can show that iterated box filtering converges to Gaussian convolution and thus to linear diffusion
$$\partial_t u = \Delta u.$$
- In a similar way, iterated median filtering approximates MCM
$$\partial_t u = \partial_{\xi\xi} u.$$
- This result is of theoretical nature. In practice, a median filter of finite pixel size usually becomes stationary after some iterations.

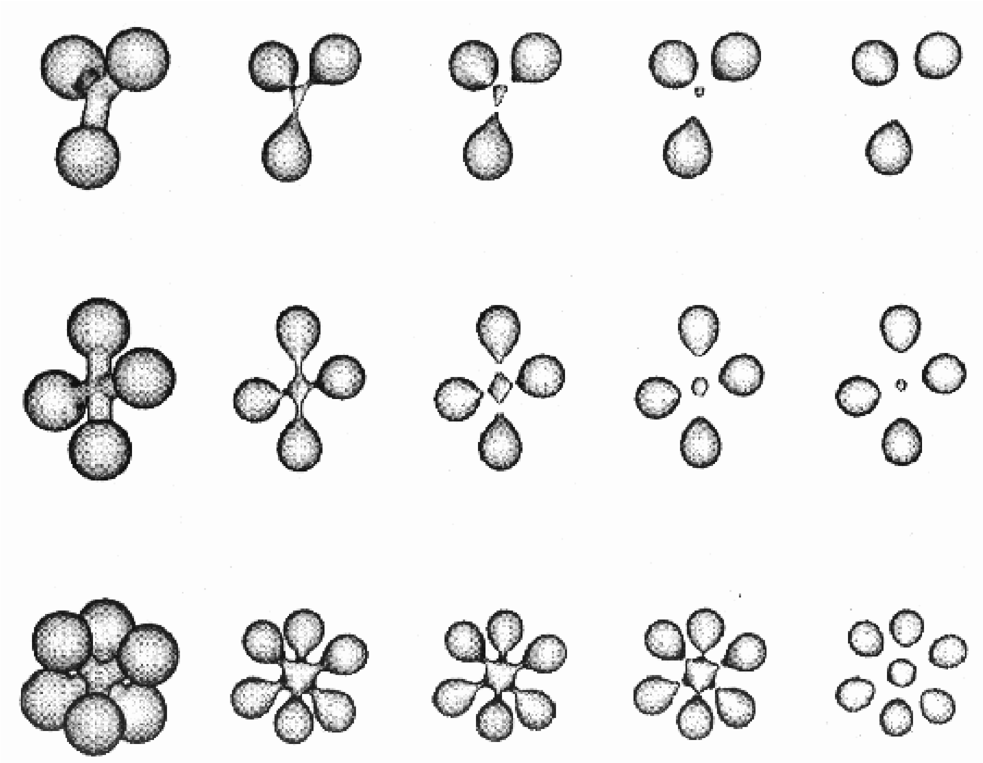
Extensions

Three-Dimensional MCM

(Chopp/Sethian 1994)

- 3-D MCM is governed by
$$\partial_t u \;=\; |\nabla u| \, \mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right)$$
where $\nabla = (\partial_x, \partial_y, \partial_z)^\top$.
- In contrast to 2-D, topological changes may occur
(reminescent of diffusion filters when going from 1-D to 2-D).

Extensions (2)



In 3-D, evolution under MCM does not necessarily preserve topology. **From left to right:** Evolutions under 3-D MCM. **Authors:** D. L. Chopp, J. A. Sethian (1994).

Extensions (3)

MCM for Vector-Valued Images

(Chambolle 1994)

- define isophote direction ξ of some vector-valued image $(u_1(x,y), ..., u_m(x,y))^\top$ as the unit eigenvector to the minimal eigenvalue of $\sum_{i=1}^m \nabla u_i \nabla u_i^\top$
- consider evolution

$$\partial_t u_i \;=\; \partial_{\xi\xi}\, u_i \qquad (1 \le i \le m)$$

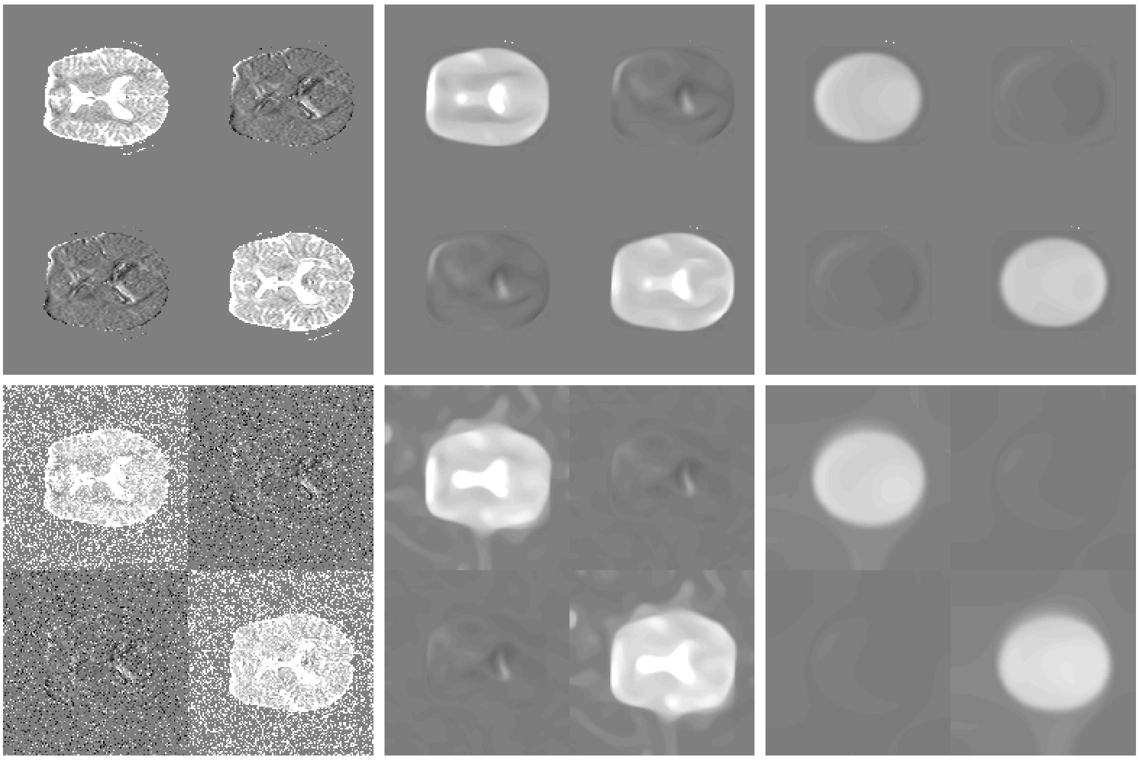
- couples all channels via a joint smoothing direction (cf. Lecture 14)

MCM for Matrix-Valued Images

(Feddern et al. 2006)

- define isophote direction ξ of some matrix-valued image $(u_{i,j}(x,y))$ as the unit eigenvector to the minimal eigenvalue of $\sum_{i,j} \nabla u_{i,j} \nabla u_{ij}^\top$
- use evolution

$$\partial_t u_{i,j} \;=\; \partial_{\xi\xi}\, u_{i,j} \qquad (1 \le i,j \le m)$$



Tensor-valued mean curvature motion. **Top row, from left to right:** Original tensor image of size 256×256, at time $t = 24$, at time $t = 240$. **Bottom row, from left to right:** Same experiment with 30 % noise.

Application: Classification of Chrysanthemum Leaves

(Abbasi/Mokhtarian/Kittler 1997)

Problem:

- British Plant Breeders Right grants exclusive right to sell new varieties for some time period.
- A new Chrysanthemum variety must differ in at least one feature from 3000 registered varieties.
- so far: manual inspection, heuristic criteria, time consuming

Goal:

- semi-automatic classification showing a small set of similar existing varieties

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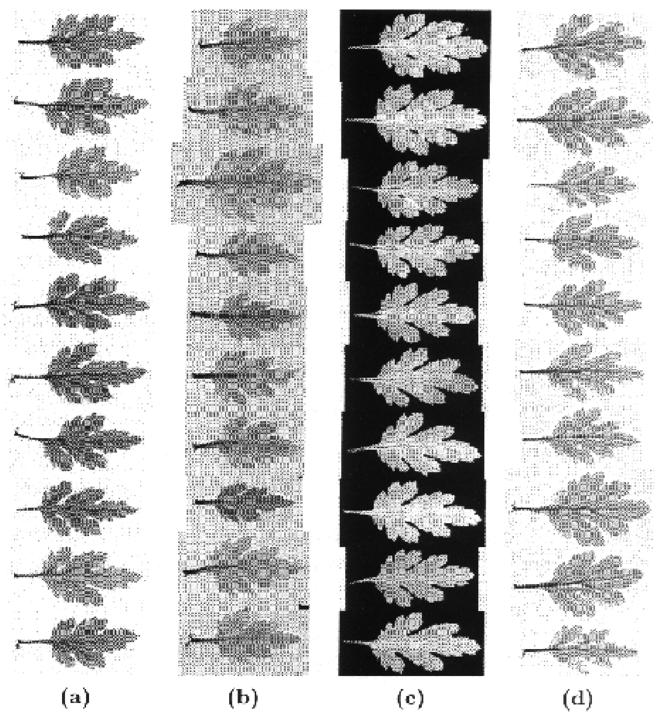
Application: Classification of Chrysanthemum Leaves (2)



Four different classes of Chrysanthemum leaves. Note the large variability within each class. **Authors:** Abbasi et al. (1997).

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Application: Classification of Chrysanthemum Leaves (3)

Method:

Step 1: Feature Extraction

- thresholding
- contour extraction and arc-length parametrisation
- study location of curvature zero-crossings in MCM scale-space
- location of significant toppoints gives feature vector

Step 2: Classification

- compensate for orientation differences by cyclic shift
- sum up Euclidean distances between corresponding toppoints
- show five closest varieties

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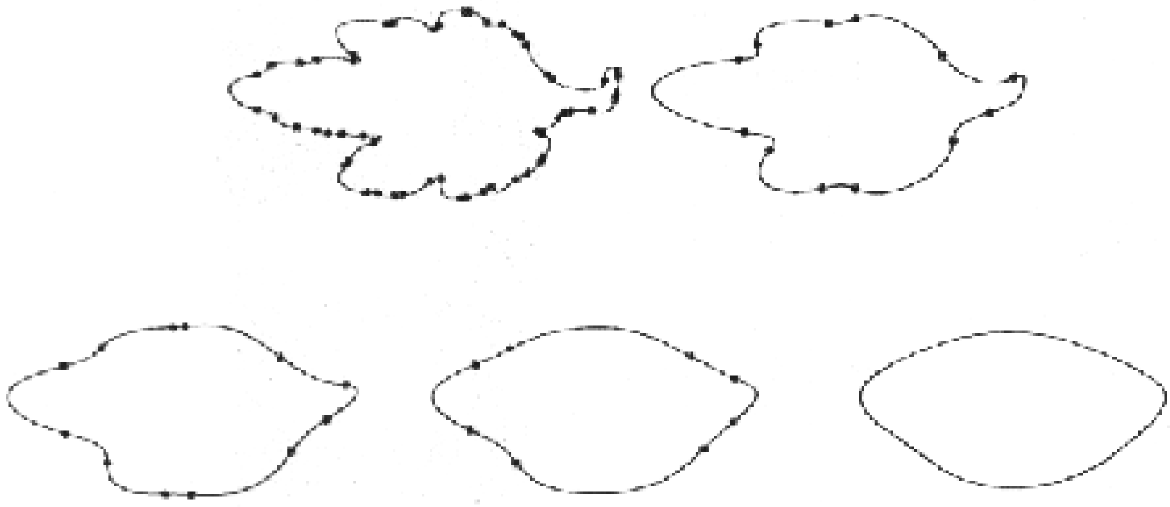
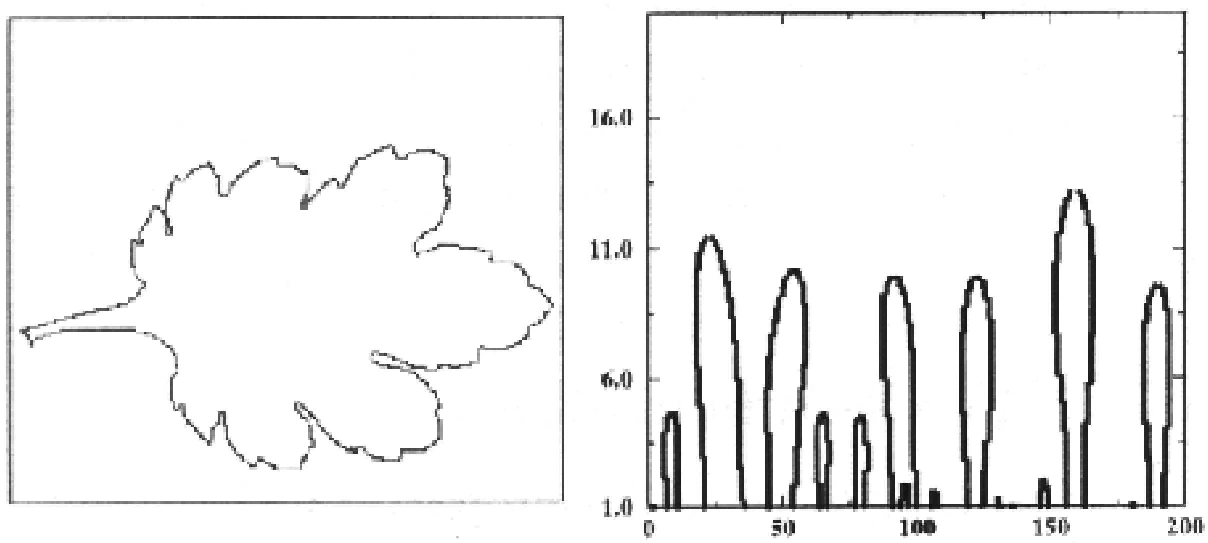
Application: Classification of Chrysanthemum Leaves (4)



Curve evolution under the MCM scale-space reduces the number of curvature zero-crossings. **Authors:** Abbasi et al. (1997).

Application: Classification of Chrysanthemum Leaves (5)



Left: Contour of a Chrysanthemum leaf. **Right:** Corresponding curvature zero-crossings in MCM scale-space. The horizontal axis depicts the normalized arc-length, and the vertical axis denotes scale. The points with horizontal tangent (toppoints) create the feature vector. **Authors:** Abbasi et al. (1997).

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Performance

- The correct variety is among the proposed five varieties in up to 95 % of all cases.
- computing time: 2 seconds (1997)

Another Application of MCM Scale-Space

(Mokhtarian/Bober 2003)

- It has been selected as a contour shape descriptor for the MPEG-7 movie compression standard.

Summary

- studied mean curvature motion (MCM) as a prototype for a curvature-driven morphological processes:
- well-posed scale-space evolutions
- cannot enhance edges
- no creation of extrema in 2-D, but topological changes in 3-D
- MCM can also be generalised to the vector- and matrix-valued case.
- useful for shape simplification tasks

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Assignment T6 – Theoretical Home Work

Problem 1 (Discretisation of Hyberbolic PDEs) (3 points)

Let the following simple hyperbolic PDE be given:

$$u_t = a u_x \,,$$

where is $a \in \mathbb{Z}$.

(a) Discretise this PDE in three ways: by using forward, backward and central differences.

(b) What can you state about the min-max-stability of the three resulting schemes? What is the influence of the sign of a?.

Problem 2 (Slope Transform) (4 points)

The slope transform of a 1-D signal $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathcal{S}[f](u) \;:=\; \operatorname*{stat}_x (f(x) - ux)$$

where $\operatorname{stat}_x g(x) := \{g(x) | g'(x) = 0\}$ denotes the set of stationary values of a function $g(x)$. Let the functions $f_p : \mathbb{R} \to \mathbb{R}$ be given by $f_p(x) := cx^p$ for $p = 2, 4, 6, \ldots$ with a constant $c > 0$. Compute the slope transform $\mathcal{S}[f_p]$ of f_p. What special property do you notice for $p = 2$?

Assignment T6 – Theoretical Home Work

Problem 3 (Curvature-Based Morphology) (3 points)

Prove the following equivalences for MCM:

$$\begin{aligned}
\partial_t u &= \partial_{\xi\xi} u \\
&= \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2} \\
&= \Delta u - \frac{1}{|\nabla u|^2} \nabla u^\top \mathrm{Hess}(u) \nabla u \\
&= |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)
\end{aligned}$$

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Assignment T6 (3)

Problem 4 (Multiple Choice) (2 points)

Answer the following questions with YES or NO. You get 0.5 points per correct answer and lose one 0.5 points per wrong answer. You cannot get less than 0 points.

(a) Let us consider the corresponding semi-implicit scheme from Lecture 19, Slide 5. The extension of this scheme for 3-D optic flow computation requires the solution of three pentadiognal systems of equations each step.

(b) Gaussians are invariant under the slope transform.

(c) Mean curvature motion preserves the average grey value.

(d) The shape-inclusion principle for affine morphological scale-space only holds for ellipses.

Deadline for submission: Friday, July 4, 10 am (before the lecture).